Controllability analysis of multi-agent systems using relaxed equitable partitions

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Abstract: This paper investigates how to make decentralised networks, amenable to external control, i.e., how to ensure that they are appropriately organised so that they can be effectively 'reprogrammed'. In particular, we study networked systems whose interaction dynamics are given by a nearest-neighbour averaging rule, with one leader node providing the control input to the entire system. The main result is a necessary and sufficient condition for the controllability of such systems in terms of the graph topology. In particular, we give a graph theoretic interpretation of the controllability properties through the so-called relaxed equitable partition.

Keywords: networked control systems; network analysis and control; communication networks.

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1 Introduction

The emergence of decentralised, mobile multi-agent networks, such as distributed robots, mobile sensor networks, or mobile ad-hoc communications networks, has imposed new challenges when designing control algorithms. These challenges are due to the fact that the individual agents have limited computational, communications, sensing, and mobility resources. In particular, the information flow between nodes of the network must be taken into account explicitly already at the design phase, and a number of approaches have been proposed for addressing this problem, e.g., Jadbabaie et al. (2003), Ji and Egerstedt (2007), Lin et al. (2005), Martinez et al. (2007), Mesbahi (2005), Olfati-Saber et al. (2007), Tanner et al. (2007), and Cortes et al. (2006). There is an active research effort underway in the control and dynamical systems community to formalise these systems and lay out a foundation for their analysis and synthesis. The problem of the controllability of a system comprising a large number of autonomous agents in the sense that we aim to characterise conditions under which some control stations can move the agents to any desired position, has attracted substantial attention and has been largely studied over the past years. In this direction (Kobayashi et al., 1978) represents one of the first papers focused on the controllability issue for decentralised information structures. A general decentralised systems with two control stations was considered and a necessary and sufficient condition for controllability was given by combining the informations obtained from both the controllability and the observability matrices. By following this path, (Anderson and Moore, 1981) provided general rules for designing feedback laws that can be used to stabilise unstable decentralised systems. As a result of this effort, over the past few years, a distinct area of research that lies at the intersection of systems theory and graph theory has emerged. This intersection was first introduced in Corfmat and Morse (1976), where concepts from graph theory and from geometric theory of linear systems were used to derive explicit conditions for determining when the closed-loop spectrum of a multi-channel linear system can be stabilised with decentralised control, and has been further developed for example in Gong and Aldeen (1997) where a necessary and sufficient condition for the existence of a decentralised controller which stabilises a system is stated in terms of fixed modes of a quotient system identified using graph theory concepts.

Regardless of whether the information flow is generated over communication channels or through sensory inputs, the underlying geometry is playing an important role. For example, if an agent is equipped with omnidirectional range sensors, it can only detect neighbouring agents if they are located in a disk around the agent. Similarly, if the sensor is a camera, the area becomes a wedge rather than a disk. But, to make the interaction geometry explicit when designing control laws is not an easy task, and an alternative view is to treat interactions as purely combinatorial. In other words, all that matters is whether or not an interaction exists between agents, and under certain assumptions on the global interaction topology, one can derive remarkably strong and elegant results. (For a representative sample, see Jadbabaie et al., 2003; Olfati-Saber et al., 2007; Tanner et al., 2007). What then remains to be shown is that the actual geometry in fact satisfies the combinatorial assumptions.

In this paper, we continue down this path, by investigating controllability from a graph-theoretic point-of-view, which was first proposed in Tanner (2004), and later investigated in Rahmani et al. (2009). In Rahmani et al. (2009), necessary conditions for controllability were given entirely in terms of the graph topology and, as such, it provides a starting point for the undertakings in this paper. In particular, we show that when the network is not completely controllable, the controllable subspace can be given a graph-theoretic interpretation. What this means is that it is possible to construct a smaller, completely controllable network (the so-called *controllable relaxed quotient graph*) that is equivalent to the original network in terms of controllable subspaces. This design allows the control designer to focus directly on a smaller network when producing control laws.

Moreover, it is shown that the dynamics associated with the uncontrollable part of the network is asymptotically stable for all connected networks. As such, the controllable relaxed quotient graph is an *approximate bisimulation* of the original network, in the sense of Girard and Pappas (2007).

Furthermore, the topic on how to overcome the uncontrollability issue in single leader symmetric networks is then considered. It is shown that by associating different weights to nodes, we can modify controllability properties of the network. In particular, this can be used as a way of breaking existing symmetry-induced lack of controllability.

Following this approach, the study of the controllability for single-leader systems is then continued via tools from algebraic graph theory. Our aim is to find a method for giving a direct interpretation of the controllability of the network from a graph-theoretic vantage point. In this sense, we provide a necessary and sufficient graphical condition through relaxed equitable partition concepts for the system's controllability.

Although the main results in this paper provide topological characterisations of controllability for networked systems, it must be pointed out that it is at this point not clear if the methods based on relaxed equitable partitions are computationally to prefer to the standard rank test. However, the contribution of this paper should be understood as expanding our understanding of the role that the network topology plays when designing decentralised networked system modelled with the consensus agreement and the graph-Laplacian theory. In particular, the establishment of a network interpretation of controllability is useful for incrementally building networks by avoiding uncontrollability situations directly at the design phase, without having to explicitly compute other mathematical structures such as the controllability matrix.

The outline of this paper is as follows: in Section 2, we briefly review the basic premises behind leader-follower networks and recall some definitions from algebraic graph theory. In Section 3, we review some results from Rahmani et al. (2009) and Tanner (2004), allowing us to study controllability of single-leader networks from a graph-theoretic vantage-point. Relaxed Quotient graphs, obtained through so-called *relaxed equitable partitions* of the graph, are the topic of Section 4, while the uncontrollable part of network is discussed in Section 5. Important results of this paper are given in Section 6, followed by simulations in Section 7. In Section 8, we investigate and we solve the problem of the uncontrollability of symmetric single networks. The main results of this paper are in Section 9 where we provide a necessary sufficient condition for single leader networks using relaxed equitable partitions, followed by examples in Section 10.

2 Leader follower consensus networks

In multi-agents systems, it is common to let the nodes of a graph represent the agents, and to let the arcs in the graph represent the inter-agent communication links. In fact, this interaction graph plays a central role in representing the information flow among the agents, and in defining the properties of the system.

Let the undirected graph G be given by the pair $(V, \mathcal{E})$, where $V = \{1, \ldots, n\}$ is a set of n vertices, and $\mathcal{E}$ is a set of edges. We can associate the *adjacency matrix* $\mathcal{H} \in \mathbb{R}^{n \times n}$ with G, whose entries satisfy

$$[\mathcal{H}]_{ij} = \begin{cases} 1 & \text{if } (i,j) \in \mathcal{E} \\ 0 & \text{otherwise.} \end{cases}$$

Two nodes j and k are neighbours if $(j,k) \in \mathcal{E}$, and the set of the neighbours of the node j is defined as $N_j = \{k \,|\, [\mathcal{H}]_{jk} = 1\}$. The degree of a node is given by the number of its neighbours, and a graph G is *connected* if there is a path between any pair of distinct nodes, where a path $i_0 i_1 \ldots i_S$ is a finite sequence of nodes such that $i_{k-1} \in N_k$ with $k = 2, 3 \ldots S$.

In this paper we let the state of each node, x_i, be scalar. (This does not affect the generality of the derived results.) The standard, consensus algorithm is the update law

$$\dot{x}_i(t) = \sum_{j \in N_i} (x_j(t) - x_i(t)), \tag{1}$$

or equivalently $\dot{x}(t) = -\mathcal{L}x(t)$, where $x(t)$ is the vector with the states of all nodes at time t, and $\mathcal{L}$ is the graph Laplacian. $\mathcal{L}$ can be obtained as $\mathcal{I}\mathcal{I}^T$, where $\mathcal{I} \in \mathbb{R}^{n \times p}$, ($p$ being the number of edges), is the *incidence matrix* of the graph, defined as

$$[\mathcal{I}]_{kl} = \begin{cases} 1 & \text{if node } k \text{ is the head of the edge } l \\ -1 & \text{if node } k \text{ is the tail of the edge } l \\ 0 & \text{otherwise,} \end{cases}$$

given an arbitrarily orientation of the edges.

Under some connectivity conditions, the consensus algorithm is guaranteed to converge, i.e., $\lim_{t \to +\infty} x_i(t) = g$, $i \in \{1, \ldots, n\}$, where g is a constant depending on $\mathcal{L}$, and on the initial conditions $x_0 = x(0)$ (see for example, Jadbabaie et al., 2003; Olfati-Saber and Murray, 2003; Olshevsky and Tsitsiklis, 2006).

As in Tanner et al. (2007), Rahmani et al. (2009), and Ji et al. (2006), we imagine that a subset of the agents have superior sensing, computation, or communication abilities. We thus partition the node set V into a leader set L of cardinality n_l, and a follower set F of cardinality n_f, so that $L \cap F = \emptyset$ and $L \cup F = V$.

Leaders differ in their state update law in that they can arbitrarily update their positions, while the followers execute the agreement procedure (1), and are therefore controlled by the leaders.

Under the assumption that the first n_f agents are followers, and the last $n_l = n - n_f$ are leaders, the introduction of leaders in the network induces a partition of the incidence matrix $\mathcal{I}$ as

$$\mathcal{I} = \begin{bmatrix} \mathcal{I}_f \\ \mathcal{I}_l \end{bmatrix},$$

where $\mathcal{I}_f \in \mathbb{R}^{n_f \times p}$, $\mathcal{I}_l \in \mathbb{R}^{n_l \times p}$, and the subscripts f and l denote respectively the affiliation with the leaders and followers set. As a result, the graph Laplacian $\mathcal{L}$ becomes

$$\mathcal{L} = \left[\begin{array}{c|c} \mathcal{L}_f & \mathcal{L}_{fl} \\ \hline \mathcal{L}_{fl}^T & \mathcal{L}_l \end{array}\right],$$

with $\mathcal{L}_f = \mathcal{I}_f \mathcal{I}_f^T \in \mathbb{R}^{n_f \times n_f}$, $\mathcal{L}_l = \mathcal{I}_l \mathcal{I}_l^T \in \mathbb{R}^{n_l \times n_l}$ and $\mathcal{L}_{fl} = \mathcal{I}_f \mathcal{I}_l^T \in \mathbb{R}^{n_f \times n_l}$.

The control system we now consider is the controlled agreement dynamics (or leader-follower system), in which followers evolve through the Laplacian-based dynamics

$$\begin{aligned} \dot{x}_f(t) &= -\mathcal{L}_f x_f(t) - \mathcal{L}_{fl} x_l(t) \\ x_l(t) &= u(t), \end{aligned} \tag{2}$$

where x_f and x_l are respectively the state vectors of the followers and the leaders, and $u(t)$ denotes the exogenous control signal dictated by the leaders.

3 Controllability of single-leader networks

In this section, we recall some previous results of relevance to the developments in this paper. To conform to standard notation, we denote with $n = n_f$ the number of followers, we identify matrices A and B with $-\mathcal{L}_f \in \mathbb{R}^{n\times n}$ and $-\mathcal{L}_{fl} \in \mathbb{R}^{n\times 1}$ respectively, and we will equate x_f and x_l with x and u. Thus the system (2) becomes

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{3}$$

with controllability matrix

$$C = [B \quad AB \quad \cdots \quad A^{n-1}B]. \tag{4}$$

As A is symmetric it can be written on the form $U\Lambda U^T$, where Λ is the diagonal matrix of eigenvalues of A and U is the unitary matrix comprised of its pairwise orthogonal unit eigenvectors. Since $B = UU^TB$, by factoring the matrix U from the left in equation (4), C assumes the form

$$C = U[U^TB \quad \Lambda U^TB \quad \cdots \quad \Lambda^{n-1}U^TB].$$

If one of the columns of U is perpendicular to all the columns of B, then C will have a row equal to zero and hence be rank deficient. On the other hand, in the case of one leader, if any two eigenvalues of A are equal, then C will have two linearly dependent rows, and again, the controllability matrix becomes rank deficient, as shown in Tanner (2004). Moreover, the system is *leader symmetric* if there is a non-identity permutation J (matrix defined over the follower nodes F) such that $JA = AJ$. As shown in Rahmani and Mesbahi (2006), in that case the system (3) is uncontrollable because one of the eigenvectors of A is also orthogonal to all columns of B.

In this paper, we will focus on networks that are leader symmetric, restricted to the case when $n_l = 1$, i.e., when there only is one leader present. This is related to the necessity to use the most simple framework to control a set of followers. By moving only one 'super-node', the leader, we are able to control all the agents belonging to the network.

Definition 1 (LS^{2}L Network)**:** A connected network G is said to be LS^{2}L (Leader-Symmetric, Single Leader) if it is leader-symmetric with a single leader.

In the following we give a graph-theoretic interpretation of the controllable part of a LS^{2}L network starting from the analysis of the controllable subspace.

4 Relaxed equitable partitions and quotient graphs

To obtain the controllable quotient graphs, the notion of a *relaxed equitable partition* is needed.

Definition 2 (Relaxed Equitable Partition)**:** A partition π of V, with cells $C_1, C_2, \ldots, C_r$ is said to be relaxed equitable if each node in C_i has the same number of neighbours in C_j $\forall i, j \in \{1, \ldots, r\}$, $i \neq j$, with $r = |\pi|$, which denotes the cardinality of the partition.

Notice that our definition differs from the standard definition of an Equitable Partition (e.g., Godsil and Royle, 2001) in that we do not insist that the subgraph induced by each cell should be regular. This means that under our definition, nodes in a cell need not have the same number of neighbours inside their own cell.

The directed graph G/π with the r cells of π as its vertices and b_{ij} edges from the ith to the jth cells of π is called the *relaxed quotient graph*, and it has no self-arcs. Moreover, a partition π with at least one cell with more than one node is said to be a Nontrivial Relaxed Equitable Partition (NREP).

Definition 3 (Characteristic Vector)**:** A characteristic vector $p_i \in \mathbb{R}^{n\times 1}$ of a nontrivial cell C_i is defined as:

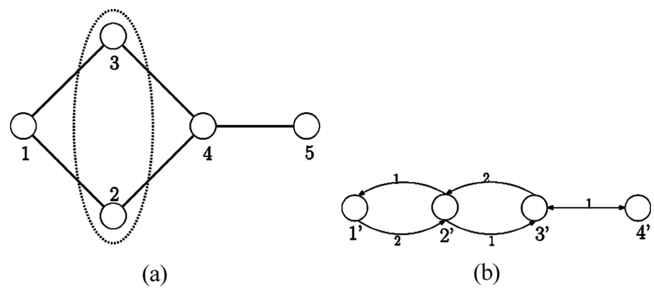
$$[p_i]_j = \begin{cases} 1 & \text{if } j \in C_i \\ 0 & \text{otherwise.} \end{cases}$$

Definition 4 (Characteristic Matrix)**:** A characteristic matrix $P \in \mathbb{R}^{n\times r}$ of a partition π of $V(G)$ is a matrix with the characteristic vectors of the cell as its columns.

An example of a nontrivial relaxed equitable partition is shown in Figure 1(a), together with its quotient graph in Figure 1(b). The characteristic matrix of this partition is given by

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Figure 1 The relaxed equitable partition $\pi = \{1', 2', 3', 4'\}$ (1(a)) with $1' = \{1\}$, $2' = \{2, 3\}$, $3' = \{4\}$, $4' = \{5\}$, and its quotient graph (1(b))



Definition 5 (Leader-Invariant relaxed Equitable Partition (LEP))**:** By the Leader-Invariant relaxed Equitable Partition (LEP), we understand the maximal relaxed equitable partition $\pi_M = \pi_F \cup \pi_L$, where $\pi_F = \{C_1^M, C_2^M, \ldots, C_s^M\}$ is the maximal relaxed equitable partition of followers such that the cardinality of π_F is minimal (i.e., has the fewest cells), and the leader L belongs to the singleton cell $C_{s+1}^M = \{L\}$ of the partition $\pi_L = \{C_{s+1}^M\}$.

5 Controllability decomposition

We first recall the concepts of the Kalman decomposition for controllability. Considering the system (3) of a LS^2L network, we construct the controllability matrix (4) and, as previously discussed, we know that it is rank deficient. The controllability subspace, is equal to the range space of C, $(\mathcal{R}(C))$, and $\text{rank}(C)$ defines the dimension of this subspace.

Consider now any basis for this subspace. Let $d = \text{rank}(C)$ and let $(p_1, p_2, \ldots, p_d)$ be the orthogonal, unit length vectors of this basis. We can now use these vectors to obtain the first d columns of the transformation matrix $\mathcal{T}$:

$$\mathcal{T} = \left[p_1 \mid p_2 \mid \ldots \mid p_d \mid \ldots\right].$$

As $\mathcal{T}$ must be an $n \times n$ square matrix, we use then $n - d$ orthogonal, unit length vectors of the basis belonging to subspace $\mathcal{R}^{\perp}(C)$ to produce $\mathcal{T}$. Let $(p_{d+1}, p_{d+2}, \ldots, p_n)$ be these vectors. We thus have

$$\mathcal{T} = \left[\underbrace{p_1 \mid p_2 \mid \cdots \mid p_d}_{\substack{\text{basis of controllable} \\ \text{subspace}}} \quad \underbrace{p_{d+1} \mid p_{d+2} \mid \ldots \mid p_n}_{\substack{\text{basis of the complement of} \\ \text{the controllable subspace}}} \right]$$

which is non singular, producing the following system:

$$\dot{\bar{x}} = \overline{A}\bar{x} + \overline{B}u, \tag{5}$$

where

$$\overline{A} = \mathcal{T}^{-1} A \mathcal{T} = \begin{bmatrix} \overline{A}_c & 0 \\ 0 & \overline{A}_{uc} \end{bmatrix} \tag{6}$$

$$\overline{B} = \mathcal{T}^{-1} B = \begin{bmatrix} \overline{B}_c \\ 0 \end{bmatrix} \tag{7}$$

and

$$\bar{x} = \mathcal{T}^{-1} x = \begin{bmatrix} \bar{x}_c \\ \bar{x}_{uc} \end{bmatrix}. \tag{8}$$

Here the subscripts c and uc refer to the controllable and uncontrollable parts respectively.

The reason why $\overline{A}$ in the decomposition equation (6) takes on this form, i.e., that $\overline{A}$ is block diagonal, follows directly from the fact that $A = A^T$ and $\mathcal{T}$ is orthonormal, i.e.,

$$\overline{A}^T = (\mathcal{T}^{-1}\ A\ \mathcal{T})^T = \mathcal{T}^{-1} A^T \mathcal{T} = \mathcal{T}^{-1} A \mathcal{T} = \overline{A}.$$

As a result, we can decouple the system into two different subsystems, namely

$$\dot{\bar{x}}_c = \overline{A}_c \bar{x}_c + \overline{B}_c u \tag{9}$$

for the controllable part of the network, and

$$\dot{\bar{x}}_{uc} = \overline{A}_{uc} \bar{x}_{uc} \tag{10}$$

for the uncontrollable part.

Proposition 1: *Let G be a single leader network with dynamics described by equation* (3). *Its uncontrollable subsystem* (10) *is always asymptotically stable, i.e.,*

$$\lim_{t\to\infty} \bar{x}_{uc}(t) = 0.$$

Proof: Since we apply a similarity transformation $\mathcal{T}$ to A, this does not change its eigenvalues. So we need to prove that A is negative definite, which follows from the fact that $\mathcal{L}_f$ is positive definite, as shown in Ji et al. (2006). □

Proposition 2 (Range space of C)**:** *Let G be a LS^2L network with dynamics described by equation* (3)*, and let π_M be its LEP. The range space of C corresponds to the spanning set of the characteristic vectors of π_F, i.e.,*

$$\mathcal{R}(C) = \text{span}\left\{ \begin{bmatrix} E_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ E_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \cdots \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ E_s \end{bmatrix} \right\}$$

where E_i is a column vector of ones, with $r_i = |C_i^M|$ components.

Proof: We denote by r_i the cardinality of each set C_i^M of the partition π_F of G, and we now consider the graph G' in which the first r_1 vertices belong to C_1^M, the second r_2 vertices belong to C_2^M, and so on. Let $\mathcal{L}'$ be the graph Laplacian of G' and $P(G/\pi_F) \in \mathbb{R}^{n\times r}$ be the characteristic matrix of G/π_F. Recalling Definition 4, in this case we have

$$P(G/\pi_F) = \begin{bmatrix} E_1 & 0 & & 0 \\ 0 & E_2 & & 0 \\ 0 & 0 & \cdots & \vdots \\ \vdots & \vdots & & 0 \\ 0 & 0 & & E_s \end{bmatrix} \tag{11}$$

where $E_i \in \mathbb{R}^{r_i\times 1}$ is a vector with ones in each position. Now, since $A' = -\mathcal{L}'_f$ is symmetric it can be rewritten as a block matrix:

$$A' = \begin{bmatrix} A'_{11} & A'_{12} & \cdots & A'_{1,s} \\ A'_{21} & A'_{22} & \cdots & A'_{2,s} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ A'_{s,1} & \cdots & \cdots & A'_{s,s} \end{bmatrix}, \tag{12}$$

where each diagonal submatrix $A'_{ii} \in \mathbb{R}^{r_i\times r_i}$ represents the set C_i^M, and each other submatrix $A'_{ij} \in \mathbb{R}^{r_i\times r_j}$ represents the connections between nodes belonging to set C_i^M and C_j^M. From Definition 2, for each submatrix A'_{ij}, we have

$$\sum_{k=1}^{r_j} a_{i^\star k} = \sum_{k=1}^{r_j} a_{j^\star k} \quad \forall\ i^\star, j^\star \in C_i^M. \tag{13}$$

Moreover, $B' = -\mathcal{L}'_{fl}$ has always the form

$$B' = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ E \end{bmatrix} \tag{14}$$

with E column vectors of ones with l elements, where l denotes the number of the neighbours of the leader. The controllability matrix can thus be recursively calculated as

$$C = \begin{bmatrix} B' & A' \cdot B' & A' \cdot A'B' & \cdots & A' \cdot A'^{(n-2)}B' \end{bmatrix}, \tag{15}$$

and, recalling that the row sum of blocks A_{ij} are constant, it becomes

$$C = \begin{bmatrix} 0 & 0 & \cdots & \widehat{C}_{1n} \\ \vdots & \vdots & \cdots & \widehat{C}_{2n} \\ 0 & \vdots & \cdots & \vdots \\ E & \widehat{C}_{s2} & \cdots & \widehat{C}_{sn} \end{bmatrix} \quad \begin{matrix} \widehat{C}_{ij} = f_{ij}E_i \\ f_{ij} \in \mathbb{R} \end{matrix}. \tag{16}$$

So the range space of C equation (16) is such that

$$\mathcal{R}(C) = \text{span}\left\{ \begin{bmatrix} E_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ E_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \cdots \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ E_s \end{bmatrix} \right\}$$

which corresponds to the spanning set of the characteristic vectors of π_F, which proves the proposition. □

Corollary 1: *The dimension of the controllable subspace of the network G is equal to the cardinality of* π_F *of its LEP.*

Proof: The range space of C is equal to the spanning set of the characteristic vectors of π_F. It follows that $\dim(\mathcal{R}(C))$, is equal to the number of the columns of its characteristic matrix, i.e., the number of sets of π_F. □

Remark 1: As shown in Rahmani and Mesbahi (2006), if the network is LS^2L, it is not completely controllable. Then the LEP is nontrivial, i.e., not all cells are singletons.

Corollary 2: *Agents of the network belonging to each set* C_i^M *of* π_F *starting from the same point will move together, i.e.,* $\forall\ t > 0$,

$$\left.\begin{matrix} x_1(0) = \cdots = x_{r_1}(0) \\ \vdots \\ x_{n-r_s}(0) = \cdots = x_n(0) \end{matrix}\right\} \Rightarrow \left\{\begin{matrix} x_1(t) = \cdots = x_{r_1}(t) \\ \vdots \\ x_{n-r_s}(t) = \cdots = x_n(t) \end{matrix}\right. .$$

Proof: This fact follows directly from the invariance of the controllable subspace. However, it is interesting to derive the same result in a different way, which illustrates more directly its significance. A possible choice for $\mathcal{R}(C)^{\perp}$ is to take vectors with column sums to zero and with blocks $P_i \in \mathbb{R}^{r_i \times (r_i - 1)}$ in the position associate to each block E_i of $\mathcal{R}(C)$, such that

$$P_i = \begin{bmatrix} I_{r_i-1} \\ -\mathbf{1}^T \end{bmatrix}_{r_i \times (r_i - 1)}.$$

In other words we have that $\mathcal{R}(C)^{\perp} = \bigcup_{i=1}^{s} R_i$ where

$$R_1 = \text{span}\left\{ \begin{bmatrix} E_{11}^{\perp} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} E_{12}^{\perp} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots \begin{bmatrix} E_{1r_1-1}^{\perp} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right\}$$

$$R_2 = \text{span}\left\{ \begin{bmatrix} 0 \\ E_{21}^{\perp} \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ E_{22}^{\perp} \\ \vdots \\ 0 \end{bmatrix}, \cdots \begin{bmatrix} 0 \\ E_{1r_2-1}^{\perp} \\ \vdots \\ 0 \end{bmatrix} \right\}$$

and so on, where

$$E_{i1}^{\perp} = \begin{bmatrix} P \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad E_{i2}^{\perp} = \begin{bmatrix} 0 \\ P \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots \quad \text{with } P \in \mathbb{R}^{2\times 1} \quad \text{s.t.} \quad P = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

It follows that for every block E_i, i.e., for every set C_i^M, we have, $\forall t > 0$

$$\left.\begin{array}{c} x_1(0) = \cdots = x_{r_1}(0) \\ \vdots \\ x_{n-r_s}(0) = \cdots = x_n(0) \end{array}\right\} \Rightarrow \left\{\begin{array}{c} x_1(t) = \cdots = x_{r_1}(t) \\ \vdots \\ x_{n-r_s}(t) = \cdots = x_n(t) \end{array}\right.$$

which proves the corollary. □

6 Approximate bisimulation through relaxed equitable partition graph

A common theme in the theory of distributed processes and in systems and control theory is to characterise systems which are ‘externally equivalent’. The intuitive idea is that we only want to distinguish between two systems if the distinction can be detected by an external system interacting with these systems. This is a fundamental notion in design, allowing us to switch between externally equivalent representations of the same system and to reduce subsystems to externally equivalent but simpler subsystems.

A crucial notion in this sense is the concept of bisimulation. The notion of bisimulation, introduced in Milner (1989), and which has been further developed for example, in Haghverdi et al. (2002), Girard and Pappas (2005), and Girard et al. (2006), is one such formal notion of abstraction that has been used for reducing the complexity of finite state systems and expresses when a subprocess can be considered to be externally equivalent to another (hopefully simpler) process.

Bisimulation is a concept of equivalence that has become a useful tool in the analysis of concurrent processes. It also reflects classical notions in systems and control theory such that state – space equivalence of dynamical systems, and especially the reduction of a dynamical system to an equivalent system with minimal state – space dimension.

In the following we apply concepts of approximate bisimulations to multi agent systems. We aim to find a subgraph of the original graph that we can use to move all the agents belonging to the network, and we aim to give a graphic and immediate interpretation to this one using relaxed equitable partitions. Indeed, since the uncontrollable part of the system is always asymptotically stable, we can simplify the original network with one which corresponds exactly to the controllable part of the network. In order to move all the agents of the network, it is possible to control this smaller entity and ignoring the uncontrollable part. Moreover, we will prove that this controllable subgraph can be found by investigating the network through relaxed equitable partitions.

Consider the controllability decomposition (6)–(8) with

$$\mathcal{T} = [\mathcal{T}_c \,|\, \mathcal{T}_{uc}] = [\mathcal{T}_c^1 \quad \mathcal{T}_c^2 \quad \cdots \quad \mathcal{T}_c^s \,|\, \mathcal{T}_{uc}], \tag{17}$$

$$\mathcal{T}^{-1} = \left[\frac{\mathcal{T}\mathrm{inv}_c}{\mathcal{T}\mathrm{inv}_{uc}}\right] = \begin{bmatrix} \mathcal{T}^1\mathrm{inv}_c \\ \mathcal{T}^2\mathrm{inv}_c \\ \vdots \\ \mathcal{T}^s\mathrm{inv}_c \\ \hline \mathcal{T}\mathrm{inv}_{uc} \end{bmatrix}, \tag{18}$$

where $\mathcal{T}_c$ denote the first $s = \dim(\mathcal{R}(C))$ columns of $\mathcal{T}$, and $\mathcal{T}\mathrm{inv}_c$ the first s rows of $\mathcal{T}^{-1}$.

Therefore

$$\overline{A}_c = \mathcal{T}\mathrm{inv}_c \; A \; \mathcal{T}_c, \tag{19}$$

and

$$\overline{B}_c = \mathcal{T}\mathrm{inv}_c \; B, \tag{20}$$

which allows us to state the following lemma.

Lemma 1: *Let G be a* LS^2L *network, with dynamics described by equation* (3), *and let* π_M *be its LEP. If* $\mathcal{T}_c$ *corresponds to the characteristic matrix of the LEP, then* $\mathcal{T}\mathrm{inv}_c$ *equation* (18) *is such that*

$$\mathcal{T}^i\mathrm{inv}_c = \frac{(\mathcal{T}_c^i)^T}{|C_i^M|}. \tag{21}$$

Proof: We know that $\mathcal{T}^{-1} = \mathcal{T}\text{inv} = (\mathcal{T}^T\ \mathcal{T})^{-1}\ \mathcal{T}^T$ with $\mathcal{T} = [\mathcal{T}_c \,|\, \mathcal{T}_{uc}]$ and, as we proved in Proposition 2, $\mathcal{T}_c$ correspond to the characteristic matrix of π_F. Since vectors of $\mathcal{T}_c$ and $\mathcal{T}_{uc}$ are orthogonal, the matrix $\mathcal{T}^\star = (\mathcal{T}^T\ \mathcal{T})$ is such that:

$$\mathcal{T}^\star = \left[\begin{array}{c|c} \mathcal{T}_c^T\ \mathcal{T}_c & 0 \\ \hline 0 & \mathcal{T}_{uc}^T\ \mathcal{T}_{uc} \end{array}\right] = \left[\begin{array}{c|c} \mathcal{T}_{11}^\star & 0 \\ \hline 0 & \mathcal{T}_{22}^\star \end{array}\right],$$

where $\mathcal{T}_{11}^\star \in \mathbb{R}^{s\times s}$ is a diagonal matrix with $[\mathcal{T}_{11}^\star]_{ii} = |C_i^M|$, and $\mathcal{T}_{22}^\star \in \mathbb{R}^{(n-s)\times(n-s)}$.

$\mathcal{T}^\star$ is a diagonal block matrix, and its inverse can be easily evaluated:

$$(\mathcal{T}^\star)^{-1} = \left[\begin{array}{c|c} (\mathcal{T}_c^T\ \mathcal{T}_c)^{-1} & 0 \\ \hline 0 & (\mathcal{T}_{uc}^T\ \mathcal{T}_{uc})^{-1} \end{array}\right] = \left[\begin{array}{ccc|ccc} \frac{1}{|C_1^M|} & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{|C_s^M|} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & & (\mathcal{T}_{uc}^T\ \mathcal{T}_{uc})^{-1} & \end{array}\right].$$

It follows that

$$\mathcal{T}^{-1} = (\mathcal{T}^\star)^{-1}\left[\begin{array}{c} \mathcal{T}_c^T \\ \hline \mathcal{T}_{uc}^T \end{array}\right] = \left[\begin{array}{c} \frac{(\mathcal{T}_c^1)^T}{|C_1^M|} \\ \frac{(\mathcal{T}_c^2)^T}{|C_2^M|} \\ \vdots \\ \frac{(\mathcal{T}_c^s)^T}{|C_s^M|} \\ \hline \mathcal{T}\text{inv}_{uc} \end{array}\right]. \tag{22}$$

Hence

$$\mathcal{T}^i\text{inv}_c = \frac{(\mathcal{T}_c^i)^T}{|C_i^M|}, \tag{23}$$

which proves the lemma. □

Theorem 1 (Controllable subspace)**:** *Let G be a LS^2L network with dynamics* (3), *and let π_M be its LEP. The controllable subspace of G corresponds to the quotient graph G/π_M.*

Proof: It is well known that $\mathcal{L} \in \mathbb{R}^{(n+1)\times(n+1)}$ is such that $\mathcal{L} = \mathcal{D} - \mathcal{H}$, where $\mathcal{D}$ is the diagonal degree matrix and $\mathcal{H}$ is the adjacency matrix. Hence $A = -\mathcal{L}_f = -(\mathcal{D}_f - \mathcal{H}_f)$, where $\mathcal{D}_f$ and $\mathcal{H}_f$ are respectively obtained by taking the first n rows and columns of $\mathcal{D}$ and $\mathcal{H}$. We have

$$\overline{A} = -(\overline{\mathcal{D}}_f - \overline{\mathcal{H}}_f) \quad \text{where} \quad \begin{cases} \overline{\mathcal{D}}_f = \mathcal{T}^{-1}\mathcal{D}_f\ \mathcal{T} \\ \overline{\mathcal{H}}_f = \mathcal{T}^{-1}\mathcal{H}_f\ \mathcal{T} \end{cases}$$

and the matrix $\overline{A}_c$ in equation (19) can be calculated as

$$\overline{A}_c = -(\overline{\mathcal{D}}_{fc} - \overline{\mathcal{H}}_{fc}), \tag{24}$$

where

$$\overline{\mathcal{D}}_{fc} = \mathcal{T}\mathrm{inv}_c \mathcal{D}_f \ \mathcal{T}_c \tag{25}$$

$$\overline{\mathcal{H}}_{fc} = \mathcal{T}\mathrm{inv}_c \mathcal{H}_f \ \mathcal{T}_c. \tag{26}$$

Since $\mathcal{T}_c$ is equal to the characteristic matrix of π_F, and $\mathcal{T}\mathrm{inv}_c = (\mathcal{T}_c^T \mathcal{T}_c)^{-1}\mathcal{T}_c^T$ satisfies (21), $\overline{\mathcal{D}}_{fc}$ results as a diagonal matrix s.t.

$$[\overline{\mathcal{D}}_{fc}]_{ii} = \frac{\sum_{k=z_i+1}^{z_j} -a_{kk}}{|C_i^M|} = \frac{\sum \deg(C_i^M)}{|C_i^M|} \quad \text{with } j = i+1, \quad z_k = \sum_{p=0}^{k-1} r_p, \quad r_0 = 0,$$

where $-a_{kk}$ is the degree of the node k and $\sum \deg(C_i^M)$ is the sum of the degree of the $|C_i^M|$ nodes belonging to the cell C_i^M. Furthermore, in the same way, each entry ij of the matrix $\overline{\mathcal{H}}_{fc}$ in equation (26) results as the sum of arcs between nodes belonging to the cell C_i^M and the nodes belonging to the cell C_j^M (we will name this integer as $|N_{ij}|$) divided by the cardinality of the cell C_i^M, i.e.,

$$[\overline{\mathcal{H}}_{fc}]_{ij} = \frac{|N_{ij}|}{|C_i^M|}.$$

Recalling Definition 2, if $i \neq j$, $[\overline{\mathcal{H}}_{fc}]_{ij} = b_{ij}$. Now, the degree of each node can be separated into two different values: the degree due to links to other nodes of the same cell (which we will name $\deg_{\mathrm{in}}$) and the degree due to links to nodes of other cells ($\deg_{\mathrm{out}}$) with $\deg = \deg_{\mathrm{in}} + \deg_{\mathrm{out}}$ and that $\sum \deg_{\mathrm{in}}(C_i^M) = |N_{ii}|$, for each diagonal entry of equation (24) we thus have

$$[\overline{A}_c]_{ii} = -\frac{\sum \deg_{\mathrm{in}}(C_i^M) + \sum \deg_{\mathrm{out}}(C_i^M)}{|C_i^M|} + \frac{|N_{ii}|}{|C_i^M|} = -\frac{\sum \deg_{\mathrm{out}}(C_i^M)}{|C_i^M|} = -\Delta(C_i^M)$$

where $\Delta(C_i^M)$ is the degree of the cell C_i^M. That follows directly from the definition of relaxed equitable partition which require that nodes inside cell C_i have the same number of neighbours inside cell C_j with $i \neq j$. Hence,

$$\overline{A}_c = -(\overline{\mathcal{D}}_{fc} - \overline{\mathcal{H}}_{fc}) = \begin{bmatrix} -\Delta(C_1^M) & b_{12} & \cdots & b_{1s} \\ b_{21} & -\Delta(C_2^M) & \cdots & b_{2s} \\ \vdots & & \ddots & \vdots \\ b_{s1} & \cdots & \cdots & -\Delta(C_s^M) \end{bmatrix}. \tag{27}$$

It follows that $\overline{A}_c$ in equation (24) corresponds to $-\mathcal{L}_f(G/\pi_M)$.

Moreover, with B as in equation (14), the decomposition (20) is such that each entry $\overline{b}_i$ of the matrix $\overline{B}_c$ satisfies

$$\overline{b}_i = \frac{\sum_{k=r_i}^{r_i + r_{(i+1)}} b_k}{|C_i^M|}$$

i.e.,

$$\bar{b}_i = \begin{cases} 1 & \text{if } C_i^M \text{ is connected to the leader} \\ 0 & \text{otherwise.} \end{cases}$$

If we define X as the number of sets $|C_i^M|$ connected with the leader, we can conclude that the matrix

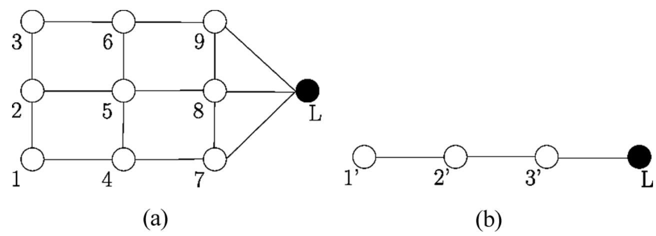
$$\left[\begin{array}{c|c} -\overline{A}_c & -\overline{B}_c \\ \hline -B_c^T & X \end{array}\right] \tag{28}$$

corresponds exactly to $\mathcal{L}(G/\pi_M)$ (defined with respect to the incidence matrix instead of the adjacency matrix, as is standard for directed graphs), which proves the theorem. □

7 A simulation study

As an application of the proposed method, consider a network consisting of nine followers and one leader. As usual, leaders and followers differ in that leaders move autonomously and 'herd' the followers, which move using the consensus protocol. Assume moreover that the followers are layed out in a grid, as in Figure 2(a). Since such structure is a LS^2L network, it is not completely controllable, and for this reason we cannot move it from any initial point to any arbitrarily point.

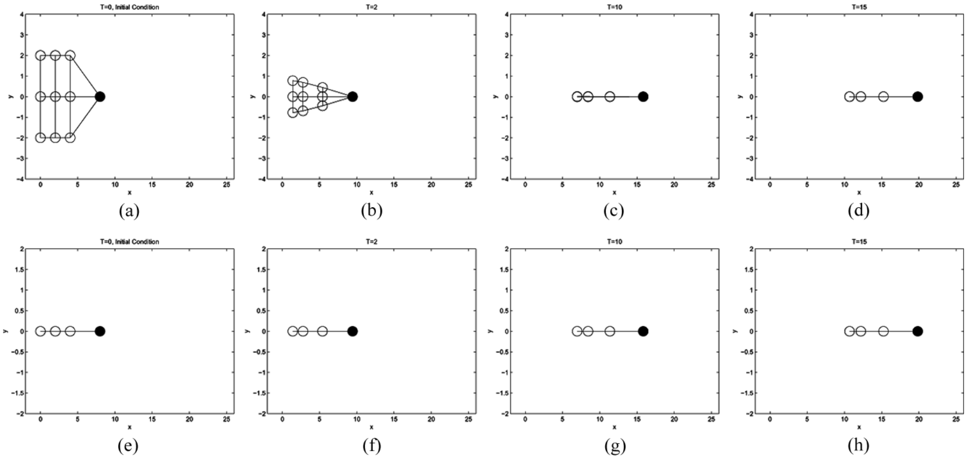
Figure 2 The graph of the network and the relaxed quotient graph corresponding to controllable part (b)



Consider now a translation of the network: due to the fact that the system is not completely controllable, this movement is not feasible. In Figure 3(a)–(d), we report some steps of a translation process of the entire network, and in Figure 3(e)–(h), we report the same steps of the same translation, but applied to its quotient graph shown in Figure 2(b).

We suppose that an external unit tells the leader the trajectory to follow, or that the leader has planning capabilities in order to solve the planning problem. Starting from the initial situation of Figure 3(a), leader moves along x axis dragging followers, whose disposition (Figure 3(c) and (d)) asymptotically converge to the controllable quotient graph (Figure 3(g) and (h)). This result emphasise the importance of a graph theoretic characterisation of the controllable part of the network, which enables the designer to focus directly on the smaller, approximate bisimulation of the original graph, when designing control laws.

Figure 3 Translation process of the entire network (a, b, c, d), and of its quotient graph (e, f, g, h)



8 Breaking the symmetry

Coordinating the agents towards a group objective is one of the most common tasks for multi-agent networks. This is a task for the leader and its ability to do so depends on the controllability properties of the networks considered. For this reason, now, given an uncontrollable network, one could ask the question of whether or not it is possible to somehow make this network controllable. This is the topic under consideration in this section, where we investigating how to change the controllability properties of LS^{2}L multi-agent systems. As previously discussed in Section 3, a LS^{2}L network is *not* completely controllable. What this means is that we cannot move nodes from any initial point to an arbitrary point (see Section 7). To overcome this, one can associate weights with the different nodes in the network that in effect means that different gains are applied at different locations, as a way of breaking the symmetry-induced lack of complete controllability. We formalise this observation in the following problem:

Problem 1 (Controllability of a LS^{2}L network)**:** *Given a LS^{2}L network G with dynamics described by equation* (3), *find a matrix* Γ *such that the new system* $\dot{x} = \Gamma Ax + \Gamma Bu$ *is completely controllable.*

It is clear that solutions of Problem 1 are not unique. However we can state the following theorem.

Theorem 2 (Breaking symmetry)**:** *Let G be a LS^{2}L network with dynamics described by equation* (3), *and let* π_M *be its LEP. Let* Γ *be a diagonal matrix of* $\gamma_1, \gamma_2, \dots, \gamma_n$ *with* $\gamma_i \neq \gamma_j$ $\forall\ i, j \in \{1, \dots, n\}$, $i \neq j$, *then the system*

$$\dot{x} = \Gamma Ax + \Gamma Bu \tag{29}$$

is completely controllable.

Proof: In Proposition 2 we proved that the range space of C corresponds to the spanning set of the characteristic vectors of π_F, i.e.,

$$\mathcal{R}(C) = \text{span}\left\{ \begin{bmatrix} E_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ E_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \cdots \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ E_s \end{bmatrix} \right\} \tag{30}$$

where E_i is a column vector of ones, with $r_i = |C_i^M|$ components.

Let now Γ be a diagonal matrix of $\gamma_1, \gamma_2, \ldots, \gamma_n$, i.e.,

$$\Gamma = \begin{bmatrix} \gamma_1 & 0 & \cdots & 0 \\ 0 & \gamma_2 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & \gamma_n \end{bmatrix}.$$

If γ's are chosen different inside each set C_i^M, i.e.,

$$\gamma_{r_i+p} \neq \gamma_{r_i+k} \quad \forall p, k \in \{1, \ldots, r_j\}, \;\; p \neq k \tag{31}$$

with

$$\begin{cases} r_0 = 0 \\ i \in \{0, 1, \ldots, r-1\} \\ j = i+1 \end{cases}$$

the controllability matrix $\mathcal{C}'$ of the system (29) become

$$C' = \begin{bmatrix} 0 & \cdots & f_{1n} \\ 0 & \ddots & \vdots \\ E & \cdots & f_{nn} \end{bmatrix} \quad \begin{matrix} f_{ij} \in \mathbb{R} \;\; \text{s.t.} \\ f_{kn} \neq f_{jn} \\ \forall k, j \in \{1, \ldots, n\}, \\ k \neq j \end{matrix}. \tag{32}$$

The reason why the controllability matrix C' is structured as in equation (32) follows because the matrix Γ breaks the constant row sum condition inside cells in the the matrix ΓA and the non-zero values of ΓB result all different. Therefore the range space of C' is

$$\mathcal{R}(C') = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \cdots \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\} \tag{33}$$

and we can conclude that the system (29) is completely controllable because $\mathcal{R}(C') = \mathbb{R}^n$. □

9 A graph characterisation of controllability of single leader networks

The problem of investigating the controllability properties of the leader-follower networks from a graph-theoretic point of view has been particularly studied in the last years. In Rahmani and Mesbahi (2006) a sufficient condition for the uncontrollability of a single leader network is given. As previously shown, if the system is leader-symmetric then the controllability matrix become rank deficient and the system is uncontrollable. Moreover, a similar controllability approach can be found in Rahmani et al. (2009), where necessary conditions are given completely in terms of the graph topology.

In this section, we continue down this path and we provide a necessary and sufficient condition for the controllability of a single leader network.

Theorem 3 (Controllability of a single leader network)**:** *Let G be a single leader network with dynamics described by equation* (3), *and let π_M be its LEP. The system is completely controllable* **if and only if** *the cardinality of π_F is equal to n, i.e., all the cells C_i^M are singleton.*

Proof of sufficiency: We use the same approach as in Proposition 2. We denote by $r_i = 1$ $(s = n)$ the cardinality of each set C_i^M of the partition π_F of G, and we now consider the graph G' in which the first r_1 vertices belong to C_1^M, the second r_2 vertices belong to C_2^M, and so on. Let $\mathcal{L}'$ be the graph Laplacian of G'. Now, since $A' = -\mathcal{L}'_f$ is symmetric it can be rewritten as a block matrix:

$$A' = \begin{bmatrix} A'_{11} & A'_{12} & \cdots & A'_{1,s} \\ A'_{21} & A'_{22} & \cdots & A'_{2,s} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ A'_{s,1} & \dots & \dots & A'_{s,s} \end{bmatrix},$$

where each diagonal submatrix $A'_{ii} \in \mathbb{R}^{1\times 1}$ represents the set C_i^M, and each other submatrix $A'_{ij} \in \mathbb{R}^{1\times 1}$ represents the connections between nodes belonging to set C_i^M and C_j^M. Since, $B' = -\mathcal{L}'_{fl}$, it has the form

$$B' = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ E \end{bmatrix} \tag{34}$$

where E is a column vector of ones with l elements, where l denotes number of the neighbours of the leader. The controllability matrix can thus be recursively calculated as

$$C = [B' \quad A' \cdot B' \quad A' \cdot A'B' \quad \cdots \quad A' \cdot A'^{(n-2)}B'],$$

and it becomes

$$C = \begin{bmatrix} 0 & \cdots & f_{1n} \\ 0 & \ddots & \vdots \\ E & \cdots & f_{nn} \end{bmatrix} \quad \begin{matrix} f_{ij} \in \mathbb{R} \text{ s.t.} \\ f_{kn} \neq f_{jn} \\ \forall k, j \in \{1, \ldots, n\}, \\ k \neq j \end{matrix}. \tag{35}$$

Hence the range space of C is

$$\mathcal{R}(C) = \text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \cdots \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \right\} \tag{36}$$

and we can conclude that the network G is completely controllable because $\mathcal{R}(C) = \mathbb{R}^n$. □

Proof of necessity: We know that equation (3) is completely controllable if and only if its controllability matrix has full rank. This means that C should be as in equation (35) and $\mathcal{R}(C) = \mathbb{R}^n$, i.e., as in equation (36).

We now define two new matrices $\widehat{A}_L \in \mathbb{R}^{n\times(n-l)}$ and $\widehat{A}_R \in \mathbb{R}^{n\times l}$ (l denotes number of the neighbours of the leader) such that $[\widehat{A}_L \ \ \widehat{A}_R] = A$, i.e.,

$$A = [\widehat{A}_L \ \ \widehat{A}_R] = \left[\begin{array}{c|c} \widehat{A}_L^1 & \widehat{A}_R^1 \\ \widehat{A}_L^2 & \widehat{A}_R^2 \\ \vdots & \vdots \\ \widehat{A}_L^n & \widehat{A}_R^n \end{array}\right]. \tag{37}$$

As usual, we evaluate the controllability matrix as

$$C = [B \quad A \cdot B \quad A \cdot AB \quad \cdots \quad A \cdot A^{(n-2)}B],$$

and it becomes

$$C = [\underbrace{\begin{matrix} 0 \\ 0 \\ \vdots \\ 1 \end{matrix}}_{C_1} \ \underbrace{\begin{matrix} \widehat{A}_R^1 E \\ \widehat{A}_R^2 E \\ \vdots \\ \widehat{A}_R^n E \end{matrix}}_{C_2} \ \underbrace{\begin{matrix} [\widehat{A}_L^1 \ \widehat{A}_R^1]\, C_2 \\ [\widehat{A}_L^2 \ \widehat{A}_R^2]\, C_2 \\ \vdots \\ [\widehat{A}_L^n \ \widehat{A}_R^n]\, C_2 \end{matrix}}_{C_3} \ \begin{matrix} \cdots \\ \\ \\ \cdots \end{matrix} \ \underbrace{\begin{matrix} [\widehat{A}_L^1 \ \widehat{A}_R^1]\, C_{n-1} \\ [\widehat{A}_L^2 \ \widehat{A}_R^2]\, C_{n-1} \\ \vdots \\ [\widehat{A}_L^n \ \widehat{A}_R^n]\, C_{n-1} \end{matrix}}_{C_n}]. \tag{38}$$

Since B is as in equation (34), and C_i, with $i \in \{1, 2, \ldots, n\}$, represents the ith column of C, it is clear that equation (38) assumes the form as in equation (35) if

$$\begin{cases} \displaystyle\sum_{k=1}^{n-l} [\widehat{A}_L]_{ik} \neq \sum_{k=1}^{n-l} [\widehat{A}_L]_{jk} \\ \displaystyle\sum_{k=n-l+1}^{n} [\widehat{A}_R]_{ik} \neq \sum_{k=n-l+1}^{n} [\widehat{A}_R]_{jk} \end{cases} \quad \forall i, j \in \{1, \ldots, n\}, \ i \neq j. \tag{39}$$

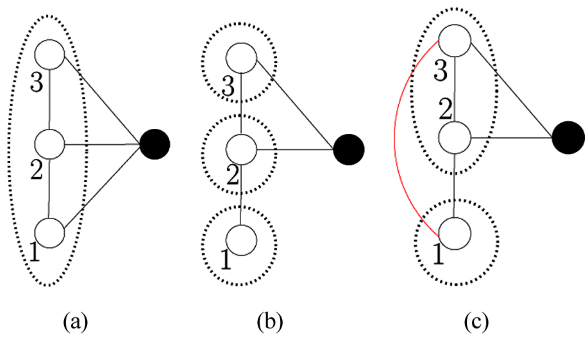
The condition (39) assures that all the cells C_i^M of the LEP of G are singletons and the theorem follows. □

Theorem 3 provides a method to identify controllability properties of multi-agent systems with a single leader. In an uncontrollable multi-agent system, nodes in the same cell are not distinguishable from the leader's point of view. This means, that agents belonging to the same cell of the LEP, if identically initialised remain undistinguished to leader throughout the system evolution (see Corollary 2). It follows that the cardinality of π_F of the LEP π_M in a completely controllable multi-agent system with a single leader is equal to n. For this reason it represents an important result in investigating the controllability of multi agent systems with a single leader. In particular it allows us to have a direct graph theoretic interpretation of the controllability of the network using relaxed equitable partition.

10 Examples

As an example consider a simple network G with three followers and only one leader (Figure 4(a)). Since the cardinality of π_F of its LEP π_M is equal to 1, the network has a controllable subspace of the same dimension. Using Relaxed Equitable Partitions concepts we can investigate on the variations of the controllability properties related to the modifications of the topology of the original graph. For this reason we first cut the edge from node 1 to the leader (Figure 4(b)); since the LEP of the new graph is composed only by singletons cell, for the Theorem 3 it is completely controllable.

Figure 4 A LS^2L network (a) and its modifications (b, c) (see online version for colours)



Consider now the graph in Figure 4(b). As the result of the connection of node 3 with node 1 (Figure 4(c)), we obtain a LEP composed by two cells (excluding that of the leader): a cell with node 3 and node 2, and a singleton cell with node 1. The controllable subspace has a dimension 2. Agents 2 and 3 if identically initialised remain undistinguished to leader throughout the system evolution and will collapse into a single node.

11 Conclusions

The problem of controllability of a group of autonomous agents has been considered. A leader-follower linear consensus network has been used to model the interactions among the nodes. It has been shown that when the network is not completely controllable, we can give a graphic theoretic interpretation to the controllability subspace, and that it is possible to construct a smaller completely controllable network that is controllable-equivalent to the original one.

Moreover, it has been shown that the uncontrollability in LS^2L networks can be overcome associating weights with the different nodes as a way for breaking the symmetry in the agents layout.

At last, it has been given a necessary a sufficient condition for the controllability of single leader multi-agent networks in term of the graph topology. In particular, a direct interpretation of the controllability properties can be given by investigating the networks through relaxed equitable partitions.

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