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CLOSED LOOP SMOOTH STEERING OF UNICYCLE-LIKE VEHICLES

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Abstract

Within this paper it is shown that, provided a special choice for the system state equations is a-priori made, the use of the simplest quadratic form as candidate Lyapunov function, directly leads to the definition of smooth and effective closed loop control laws for unicycle-like vehicles, suitable to be used for steering, path following, and navigation among assigned via points. Some considerations about the curvature of the corresponding manoeuvres are also reported.

1. Introduction

As it is well known, the majority of the mobile robots made available for research or effectively employed within real applications are, for manoeuvrability reasons, very often characterized by a unicycle-like structure. The problem of steering vehicles of such a nature has always received a great deal of attention within the wide existing literature on mobile robots (see for instance [1],[2],[3]). The research topics have also recently received a new impulse as a consequence of many of the new results obtained related with the geometric based approach to non-linear control theory [4],[5], and the emerging field of the so called non-holonomic motion planning and control [3]. Moreover, it is a matter of fact that non-holonomic systems corresponding to unicycle-like vehicles have been very often considered as a basic prototype case for testing and validating many of the techniques and algorithms developed within the above mentioned research fields.

It is however author's opinion that, at least with respect to the sole and specific case of unicycle-like vehicles steering, the generally made reference to the above mentioned set of sophisticated techniques, has always hidden what is believed to be the very simple nature of the problem, which can be handled also on the basis of a straightforward application of the widely known, and well established, Lyapunov stability theory.

More precisely, within this paper it will be explicitly brought into evidence that, provided a special choice for the system state equations is a-priori made, the analysis of a simple quadratic Lyapunov candidate function, almost trivially leads to the definition of an effective closed loop control law for the vehicle steering.

Moreover, an effective use of such control law can also be devised for both the cases of path following and navigation among (possibly on-line) assigned via points [6], without requiring, in the latter case, any sort of a-priori trajectory planning or re-planning. The paper is organized as follows: in Section 2 the basic kinematic relationships and the choice of the state variables are discussed, whereas in Section 3 the Lyapunov analysis is carried on with the support of some simulation examples. In Section 4 some comments on the curvature of the trajectories are made here again with simulation experiments. Conclusions in Section 5 will conclude the paper.

2. Kinematic Equations

Let us consider a unicycle-like vehicle initially positioned at a non-zero distance with respect to a goal frame <g>, whose motion is governed by the combined action of both the angular velocity ω and the linear velocity vector $\underline{u}$, always directed as one of the axis of its attached frame <a>, as depicted in fig. 2.1.

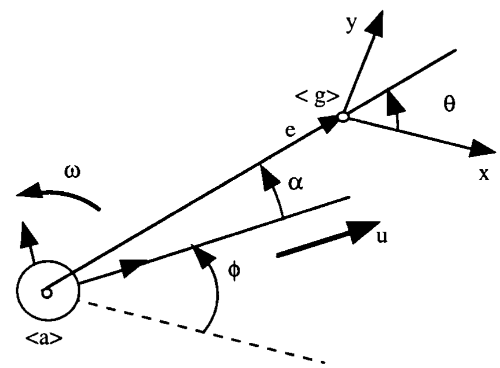

Fig 2.1

Then, one of the simplest set of kinematic equations can be directly devised, which involves the vehicle Cartesian position x, y and its orientation angle ψ, all measured with respect to the target frame-point <g>; i.e.the simple set of differential equations

$$\begin{cases} \dot{x} = u \cos \phi \\ \dot{y} = u \sin \phi \\ \dot{\phi} = \omega \end{cases} \tag{2.1}$$

being u the component of $\underline{u}$ along the vehicle principal axis.

By representing the Cartesian position of the vehicle in terms of its polar coordinates, involving the error distance e > 0 and its orientation θ with respect to <g>, and defining α

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$= \theta - \phi$ as the angle measured between the vehicle principal axis and the distance vector $\underline{e}$, the following equations are trivially obtained

$$\begin{cases} \dot{e} = -u \cos\alpha \\ \dot{\alpha} = -\omega + u \dfrac{\sin\alpha}{e} \\ \dot{\theta} = u \dfrac{\sin\alpha}{e} \end{cases} \tag{2.2}$$

Notwithstanding the fact that an infinite number of others basic kinematic equations could obviously be devised, by simply employing any non singular transformation between the various kinematic state vectors, within this work a particular attention will be however deserved to the last obtained equations (2.2), since, as it will be shown later, their form will reveal to be very suitable for easily designing appropriate closed loop control laws for the vehicle manoeuvring. As a matter of fact, equations (2.2) also result in the use of a set of state variables which are most likely resembling the same ones everyday used within our car-driving experience.

Note however that, since based on the use of polar coordinates, kinematic equations (2.2) (on the contrary of (2.1)) actually result to be valid only for non zero values assumed by the distance errors e, since both angles α and θ simply reveal to be undefined quantities in correspondence of $e = 0$; thus implying that the generally existing one-to-one correspondence with (2.1) is actually lost in correspondence of such singular points.

3. Lyapunov Function Based Closed Loop Steering

On the basis of the previous considerations concerning state equations (2.2), we are now ready to specify better the aforementioned closed loop steering problem in the following general terms:

Let the unicycle-like vehicle be initially positioned at any non zero distance from the target frame <g> and assume the state variables $[e, \alpha, \theta]'$ be directly measurable in correspondence of any $e > 0$; then find a suitable (if any) state dependent control law $[u, \omega]' = g(e, \alpha, \theta)$ which guarantees the state to be asymptotically driven to the null *limiting point* $[0, 0, 0]'$, while avoiding any attainment of the condition e=0 in finite time.

Clearly, while the former specification expresses the requirement of reaching the target frame with the appropriate orientation, the second one technically serves only for avoiding the complexities that could arise in correspondence of any finite time loss of validity of the considered model (2.2). To this respect, however note how the more general concept of "limiting point" had to be necessarily used in order to correctly specify the need of obtaining an asymptotic convergence of the state toward a point, namely the point $[0, 0, 0]'$, which is actually located on the frontier of the open set of validity of model equations (2.2) (i.e. the subset of R^3 where $e > 0$).

Theorem: There exists smooth feedback laws $u(e,\alpha,\theta)$ and $\omega(e,\alpha,\theta)$ such that any trajectory of (2.2) starting from $[e(0), \alpha(0), \theta(0)]$ with $e(0)>0$ tends to the limiting point $[0,0,0]$.

Proof. Consider the *simplest* choice for the structure of a candidate Lyapunov function related with the considered control problem; i.e. the positive definite form ($\lambda, h > 0$)

$$V = V_1 + V_2 = \frac{1}{2}\lambda e^2 + \frac{1}{2}(\alpha^2 + h\theta^2) \tag{3.1}$$

which allows for both the "error distance vector" $\underline{e}$ and the so called "alignment error vector" $[\alpha, \theta]'$ exhibited by the vehicle with respect to the target frame <g>. The time derivative $\dot{V}$ along (2.2), given by

$$\dot{V} = \dot{V}_1 + \dot{V}_2 = \lambda e \dot{e} + (\alpha \dot{\alpha} + h \theta \dot{\theta}) =$$

$$= -\lambda e u \cos\alpha + \alpha\left[-\omega + u\frac{\sin\alpha}{\alpha}\frac{(\alpha + h\theta)}{e}\right] \tag{3.2}$$

From the latter expression we can immediately see that the first term, corresponding to $\dot{V}_1$, can be made non-positive by simply letting the linear velocity u having the smooth form

$$u = (\gamma\cos\alpha)\, e \qquad ; \gamma > 0 \tag{3.3}$$

in such a way that the term $\dot{V}_1$ actually becomes

$$\dot{V}_1 = -\lambda(\gamma\cos^2\alpha)\, e^2 \leq 0 \tag{3.4}$$

Thus meaning that the first term V_1 of (3.1) is always non increasing in time and consequently, since lower bounded, asymptotically converging toward a non negative finite limit. Moreover, since V_1 is simply proportional the square of the positive scalar variable e, this fact also implies that e is monotonically non increasing in time and that the zero distance condition e=0 cannot ever be approached in finite time.

Furtherly, note also that, accordingly with the choice (3.3), the second term $\dot{V}_2$ of (3.2) becomes

$$\dot{V}_2 = \alpha\left[-\omega + \gamma\frac{\cos\alpha\sin\alpha}{\alpha}(\alpha + h\theta)\right] \tag{3.5}$$

which can be made, it also, non-positive, by simply letting the angular velocity ω taking on the smooth form (it also independent from the parameter λ, but not from h)

$$\omega = k\alpha + \gamma\frac{\cos\alpha\sin\alpha}{\alpha}(\alpha + h\theta)\ ; \quad k > 0 \tag{3.6}$$

which allows $\dot{V}_2$ to become

$$\dot{V}_2 = -k\alpha^2 \leq 0 \tag{3.7}$$

finally leading to the following expression for the time derivative of the original global Lyapunov function V

$$\dot{V} = \dot{V}_1 + \dot{V}_2 = -(\gamma\cos^2\alpha)\, e^2 - k\alpha^2 \leq 0 \tag{3.8}$$

which results in a negative semi-definite form. The uniform continuity in time of $\dot{V}$ implies the convergence of the state trajectory toward some subset of the line $[e, \alpha, \theta]' = [0, 0, \theta]'$ (i.e. toward a part of the subspace where function $\dot{V}$ can attain the null value; see (3.8)).

At this point, in order to show that the only possible convergence subset within the line $[e, \alpha, \theta]' = [0, 0, \theta]'$ is actually constituted by the sole origin point $[0, 0, 0]'$, consider the state equations (2.2) in presence of the established feedback laws (3.3), (3.6). The closed loop equations ,which are now defined also for e=0, take on the form

$$\begin{cases} \dot{e} = -(\gamma \cos^2 \alpha)\, e \\ \dot{\alpha} = -k\alpha + \gamma \dfrac{\cos\alpha \sin\alpha}{\alpha}\theta \quad ; \; e(0) > 0 \\ \dot{\theta} = \gamma \cos\alpha \sin\alpha \end{cases} \tag{3.9}$$

From (3.9) it is easily seen that the line [0,0,θ] does not contain any equilibrium point except that corresponding to the limiting point [0,0,0]'. This proves the theorem. Δ

Remark. Note that the resulting smoothness property of the obtained feedback control law, apart from the use of a Lyapunov approach, is also a direct consequence of the special choice which has been made for the state variables. Such a property is not in contrast with the result contained in the theorem of the well known paper of Brockett [8], and a detailed theoretical discussion can be found in [7]. Δ

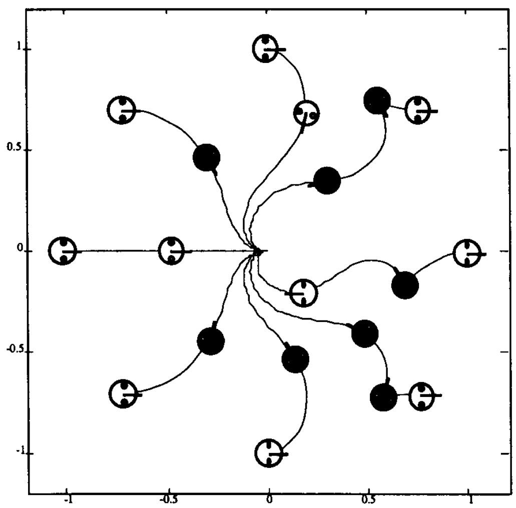


Fig. 3.1a

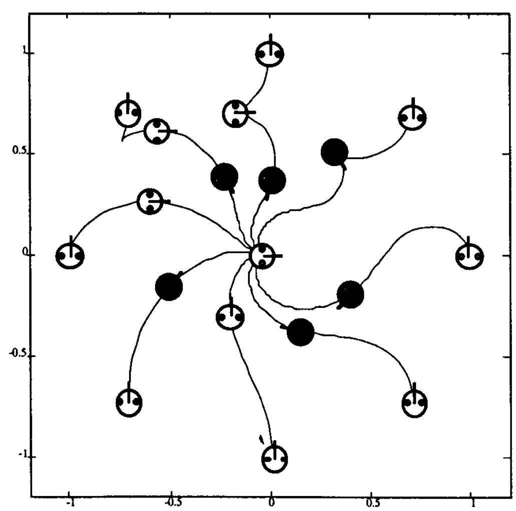


Fig. 3.1b

In the examples reported in figures. 3.1a and 3.1b, the unicycle vehicle is required to reach a final position and orientation starting from different initial configurations characterized by $e(0)=1$. More specifically, in figures 3.1a, 3.1b the initial orientations $\phi(0)$ of the vehicle with respect to the target have been assumed to be zero and $\pi/2$ respectively.

4. On the Limiting Behaviour of the Trajectory Curvature

Let us consider the resulting closed loop nonlinear equations (3.9), together with the control law expressions (3.3), (3.6) for u and ω, now considered outputs. Then, observe that in proximity of the limiting value [0, 0] for [α, θ] we can write a linear approximation, which corresponds to the following set of exponentially stable, partially decoupled, linear state equations:

$$\begin{bmatrix} \dot{\alpha} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} -k & -h\gamma \\ \gamma & 0 \end{bmatrix} \begin{bmatrix} \alpha \\ \theta \end{bmatrix} \tag{4.1}$$

$$\dot{e} = -\gamma e \tag{4.2}$$

with linear output equations

$$u = \gamma e \tag{4.3}$$

$$\omega = (k + \gamma)\alpha + h\gamma\theta \tag{4.4}$$

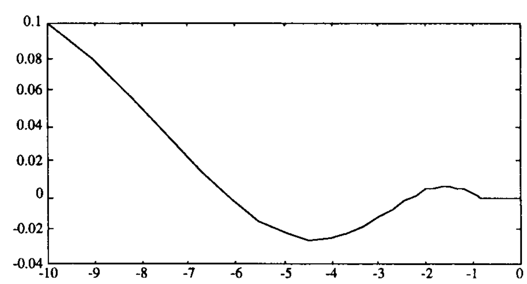
which bring into evidence the fact that, while error distance e and linear velocity u converge to zero as $\exp(-\gamma t)$, both angles α, θ and angular velocity ω converge instead as $\exp(-\sigma t)$, being $-\sigma$ the real part of the dominant pole of subsystem (4.1). At this point, we can consider the local curvature c of the vehicle trajectory; that is the ratio

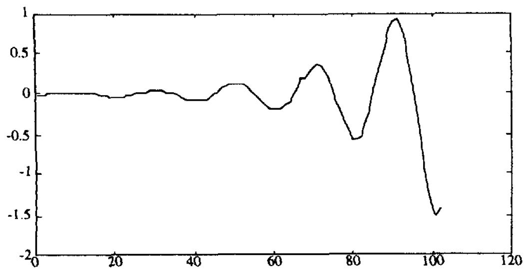
$$c = \frac{\omega}{u} \tag{4.5}$$

and consequently conclude that, for such a quantity, an *exponentially stable* convergence toward the null value is established (i.e. the vehicle approaches the target frame by asymptotically proceeding along the rectilinear path aligned with the target itself) *if and only if* $\sigma > \gamma$, or, equivalently

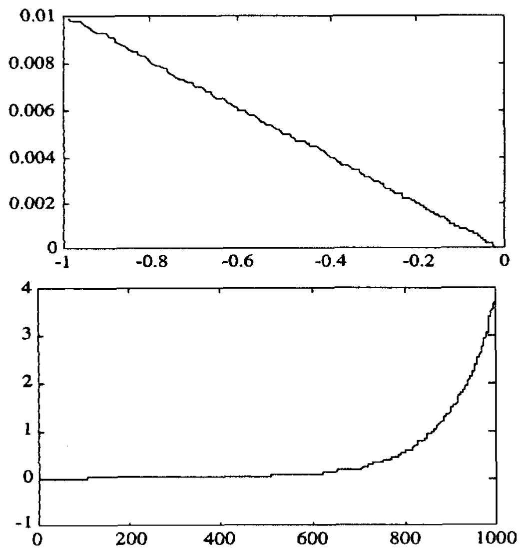
$$h > 1 \; ; \; 2\gamma < k < (h+1)\gamma \tag{4.6}$$

The following simulation examples considering the complete nonlinear model and corresponding to initial errors $e(0) \approx 10$, $\theta(0)=0.01$ and $\alpha(0) = 0.0075$, support the theoretical analysis.

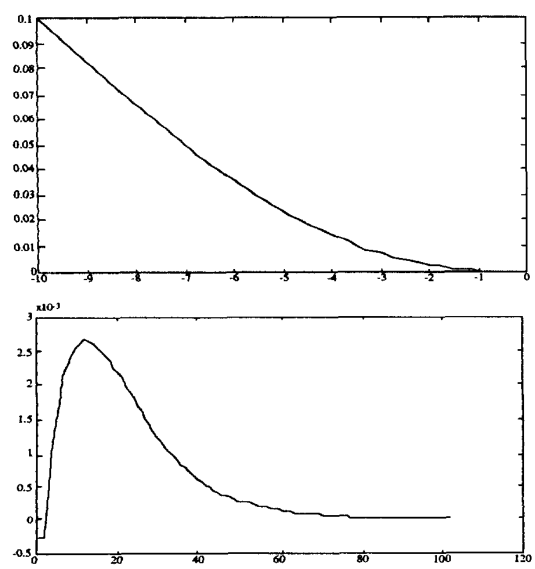




A. Trajectory and curvature for γ=1; k=1, h=50, implying σ=0.5 (system (4.1) underdamped)



B. Trajectory and curvature for γ=1; k=50, h=1, implying σ=0.02 (system (4.1) overdamped)



C. Trajectory and curvature for γ=1, k=4, h=4, implying σ=2 ((4.1) critically damped)

5. Conclusions

Within this paper it has been brought into evidence that, provided a special choice for the system state equations is a-priori made, the use of a very simple and "natural" quadratic form as candidate Lyapunov function, almost trivially leads to the definition of very simple and effective closed loop control laws for steering unicycle-like vehicles. The naturality and simplicity of the approach, whenever compared with the more sophisticated ones based on advanced non linear systems concepts and differential geometric techniques, also seems to suggest the possibility of an extension of it toward the more complex case of car like vehicles, for which extensive investigations are currently under development [7].

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