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Shortest paths for teams of vehicles*

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ABSTRACT

In this paper we consider the problem of planning motions of a team of vehicles that move in a planar environment. Each vehicle is modelled as a kinematic system with velocity constraints and curvature bounds. Vehicles can not get closer to each other than a predefined safety distance. When manipulating a common object cooperatively, further constraints apply. For such systems, we consider the problem of planning optimal paths in the absence of obstacles.

KEYWORDS: Mobile robots, Robotic teams, Shortest paths, Optimal Control, Dubins' vehicle model.

1 INTRODUCTION

In this paper we consider the problem of planning motions of a system of multiple vehicles moving in a plane. Motion of each vehicle are subject to some constraints: the velocity of the center of the vehicle is parallel to an axis fixed on the vehicle; the velocity is always nonegative along such axis; the steering radius is bounded. Also, a minimum distance between vehicles must be enforced along trajectories. When manipulating a common object cooperatively, further constraints apply. We will consider the case that cooperating vehicles share a common load (as e.g. a platform with a heavy manipulator on top), each vehicle being connected to the load through hinge joints. A bilateral constraint of constant distance between the vehicles is actually imposed in this case.

The task of each vehicle is to reach a given goal configuration from a given start configuration. Optimal solutions in the sense of minimizing total path length and total time will be considered.

The literature on optimal path planning for vehicles of this type is rather rich. The seminal work of Dubins [6] and the extension to vehicles that can back–up due to Reeds and Shepp [8], solved the single vehicle case by exploiting rather specialized tools. Later on, Sussmann and Tang [9], and Boissonnat et al. [3], reinterpreted these results as an application of Pontryagin's minimum principle ([7]). Using these tools, Bui et al. [5] performed a complete optimal path synthesis for Dubins robots. The minimum principle framework is also fundamental in the developments presented here.

Consider N vehicles in the plane, whose individual configuration is described by $\xi_i = (x_i, y_i, \theta_i) \in \mathbf{R}\times\mathbf{R}\times S^1$, with (x_i, y_i) coordinates in a fixed reference frame (o, x, y) in the plane and θ_i the heading angle of the vehicle with respect to the x axis. Each vehicle is assigned two via–point configurations, $\xi_{i,s}$ and $\xi_{i,g}$, respectively. The initial via–point time is assigned and denoted by T_i^s. Assume vehicles are ordered such that $T_1^s \leq T_2^s \leq \cdots \leq T_N^s$. We denote by T_i^g the time at which the i–th vehicle reaches its goal, and let $T_i \stackrel{def}{=} T_i^g - T_i^s$. Motions of the i–th vehicle before T_i^s and after T_i^g are not of interest.

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The i-th vehicle motion is subject to the constraint that its transverse velocity is zero, $\dot{x}_i \sin\theta_i - \dot{y}_i \cos\theta_i = 0$, $i = 1, \ldots, N$. Equivalently, this motion is described by the control system $\dot{\xi}_i = f_i(\xi_i, u_i, \omega_i)$, explicitly

$$\begin{pmatrix} \dot{x}_i \\ \dot{y}_i \\ \dot{\theta}_i \end{pmatrix} = \begin{pmatrix} u_i \cos\theta_i \\ u_i \sin\theta_i \\ \omega_i \end{pmatrix}, \tag{1}$$

where u_i and ω_i are the linear and angular velocity of the i-th vehicle, respectively. All vehicles are also supposed to be subject to the additional constraints that i) the linear velocity is unidirectional and bounded: $0 \leq u_i \leq U_i$; ii) the path curvature is bounded: $|\omega_i| \leq \Omega_i$, where $\Omega_i = \frac{|u_i|}{R_i}$ and $R_i > 0$ denotes the minimum turning radius of the i-th vehicle; iii) the distance between two vehicles must remain larger than, or equal to, a given separation limit: $V_{ij}(t) = (x_j(t) - x_i(t))^2 + (y_j(t) - y_i(t))^2 - d_{ij}^2 \geq 0$, at all times t ($d_{ii} = 0, i = 1, \ldots, N$).

The length of the path joining the viapoints for the i-th vehicle is

$$L_i = \int_{T_i^s}^{T_i^g} \sqrt{\dot{x}_i^2 + \dot{y}_i^2}\, dt = \int_{T_i^s}^{T_i^g} u_i dt \tag{2}$$

We will consider problems in which the goal is to minimize the total path length (with cost $J = \sum L_i$), and problems involving minimization of the total execution time ($J = \sum T_i$).

If separation constraints are disregarded, the minimum total length problem is clearly equivalent to N independent minimum length problems under the above constraints, i.e. to N classical Dubins' problems, for which solutions are well known in the literature ([6, 9, 3]). It should be noted that computation of the Dubins solution for any two given configurations is very efficient. If the cost to be minimized is the time employed to reach the goal T_i, the same paths result to be optimal provided they are followed by the vehicles at maximum speed $u_i = U_i$ (forward velocity would not be specified in the path length minimization problem, and actually is set to an arbitrary constant in Dubins' solution).

Consider the more general problem for multiple vehicles

$$\begin{cases} \min \sum_{i=1}^{N} J_i \\ \dot{\xi}_i = f_i(\xi_i, u_i, \omega_i) \quad i = 1, \ldots, N \\ 0 \leq u_i \leq U_i \quad i = 1, \ldots, N \\ |\omega_i| \leq \frac{|u_i|}{R_i} \quad i = 1, \ldots, N \\ V_{ij}(t) \geq 0, \quad \forall t,\ i, j = 1, \ldots, N \\ \xi_i(T_i^s) = \xi_{i,s}, \quad \xi_i(T_i^g) = \xi_{i,g},\ i = 1, \ldots, N. \end{cases} \tag{3}$$

where $J_i = L_i$ for shortest total path problems, and $J_i = T_i$ for minimum total time problems.

The shortest total path problem has a straightforward solution, consisting of the N independent Dubins' solutions, associated with any velocity profile that guarantees satisfaction of the separation constraints. For instance, execution of the Dubins' path could be sequentially scheduled so that vehicle i starts its motion after vehicle $i - 1$ has reached its goal. The existence of such a solution in all cases clearly relies on the possibility for vehicles of staying still, i.e. $u_i = 0$.

In many practical applications, however, such solution is not applicable. This is the case when the completion time for the multiple task is at a premium, as e.g. in an industrial production environment. Also, for some types of vehicles, zeroing the forward speed is not an option, as for instance in air traffic control problems. We will henceforth consider the minimum total time problem.

2 THE OPTIMAL CONTROL PROBLEM

Notice that the cost for the total time problem, $J = \sum_{i=1}^{N} T_i = \sum_{i=1}^{N} \int_{T_i^s}^{T_i^g} dt$, is not in the standard Bolza form. In order to use powerful results from optimal control theory, we rewrite the problem as follows. Let $h(t)$ denote the Heavyside function, i.e.

$$h(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases},$$

and define the window function $w_i(t) = h(t - T_i^s) - h(t - T_i^g)$. Then the minimum total time cost is written as

$$J = \int_0^\infty \sum_{i=1}^N w_i(t)dt \tag{4}$$

Using the notation $\text{col}_{i=1}^N (v_i) = \left[v_1^T, \ldots, v_N^T\right]^T$, define the aggregated state $\xi = \text{col}_{i=1}^N (\xi_i)$, controls $u = \text{col}_{i=1}^N (u_i)$ and $\omega = \text{col}_{i=1}^N (\omega_i)$, and define the ammissible control sets U and Ω accordingly. Also define the separation vector
$V = [V_{12}, \cdots, V_{1N}, V_{23}, \cdots, V_{N-1,N}]$, and the vector field $f(\xi, u, \omega) = \text{col}_{i=1}^N (f_i w_i)$. Finally introduce matrices $\Gamma_i = \text{col}_{j=1}^N (\sigma_{ij} [1\ 1\ 1]^T)$, with $\sigma_{ij} = 1$ if $i = j$, else $\sigma_{ij} = 0$, and functions $\gamma_i(\xi(t), \bar{\xi}) = \Gamma_i (\xi(t) - \bar{\xi})$. Our optimal control problem is then formulated as

Problem 1. *Minimize J subject to $\dot{\xi} = f(\xi, u, \omega)$, $u \in U$, $\omega \in \Omega$, $V \geq 0$, and to the two sets of N interior-point constraints*

$$\begin{aligned} \gamma_i(\xi(t), \xi_i^s) &= 0, \quad t = T_i^s \\ \gamma_i(\xi(t), \xi_i^g) &= 0, \quad t = T_i^g (\text{unspecified}) \end{aligned}$$

Problem 1 can be studied by adjoining the cost function with the constraints multiplied by unspecified Lagrange covectors. Necessary conditions for an extremal solution of problem 1 are obtained as:

$$\lambda_i(T_i^{s-}) = \lambda_i(T_i^{s+}) + \Gamma_i^T \pi_i^s \tag{5}$$
$$\lambda_i(T_i^{g-}) = \lambda_i(T_i^{g+}) + \Gamma_i^T \pi_i^g \tag{6}$$
$$H(T_i^{g-}) = H(T_i^{g+}) \tag{7}$$
$$\dot{\lambda}^T = \frac{\partial H}{\partial \xi} \tag{8}$$
$$\frac{\partial H}{\partial u}\delta u = 0, \quad \forall \delta u \text{ admiss.} \tag{9}$$
$$\frac{\partial H}{\partial \omega}\delta \omega = 0 \quad \forall \delta \omega \text{ admiss.} \tag{10}$$

with λ, π and ν costates of suitable dimension to be determined based on boundary conditions and Hamilton–Jacobi equations. The discussion of necessary conditions should at this point distinguish between constrained and unconstrained arcs. Along unconstrained arcs, each vehicle is subject to the same boundary conditions and dynamics of a Dubins' vehicle, hence its optimal paths will belong to Dubins' sufficient family

$$\{C_a C_b C_e \ , \ C_u S_d C_v\} \tag{11}$$

where the subscripts, indicating the length of each piece, are subject to restrictions (see e.g. [3]). Hence we have

Proposition 1 *A solution to the minimum total time problem for N vehicles that contains no constrained arcs exists if and only if for each vehicle a Dubin's trajectory exists such that no collision occur.*

2.1 Constrained arcs with 2 vehicles

To study constrained arcs, we will make the simplifying assumptions that forward velocities of all vehicles are constant and $u_i = 1$. Some further manipulation of the cost function is instrumental to deal with constrained arcs, i.e. arcs in which at least two vehicles are exactly at the critical separation ($V_{ij} = 0$, $i \neq j$). To fix some ideas, let us consider a constrained arc involving only vehicles 1 and 2. Along a constrained arc, the derivatives of the constraint must vanish:

$$N = \begin{bmatrix} V_{12} \\ \dot{V}_{12} \end{bmatrix} = \begin{bmatrix} (x_2 - x_1)^2 + (y_2 - y_1)^2 - d^2 \\ (x_2 - x_1)(\cos\theta_2 - \cos\theta_1) + (y_2 - y_1)(\sin\theta_2 - \sin\theta_2) \end{bmatrix} = 0 \tag{12}$$

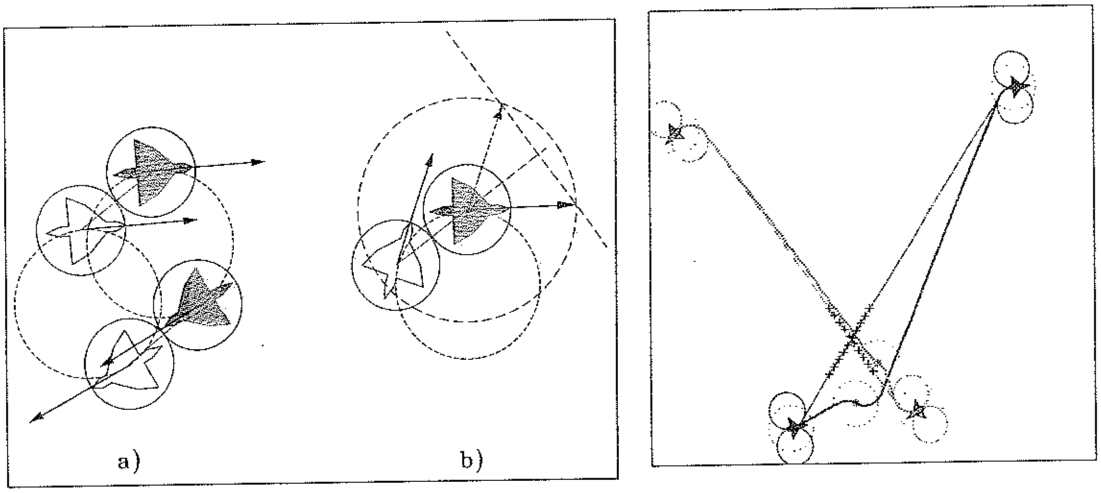

Figure 1: Left: Possible constrained arcs for two vehicles. Right: A numerically computed solution to a two–vehicles minimum total time problem. Vehicles are represented as aircraft.

with $d = d_{12}$, and

$$\ddot{V}_{12}(\xi,\omega,t) = (\dot{x}_2 - \dot{x}_1)^2 + (x_2 - x_1)(\ddot{x}_2 - \ddot{x}_1) + \quad (\dot{y}_2 - \dot{y}_1)^2 + (y_2 - y_1)(\ddot{y}_2 - \ddot{y}_1) = 0. \tag{13}$$

Let ϕ be the direction of the segment joining the two vehicles, so that

$$\begin{aligned} x_2 - x_1 &= d\cos\phi, \\ y_2 - y_1 &= d\sin\phi, \end{aligned} \tag{14}$$

From the second equation in (12), one gets

$$(x_2 - x_1)(\dot{x}_2 - \dot{x}_1) + (y_2 - y_1)(\dot{y}_2 - \dot{y}_1) = 0, \tag{15}$$

and, using (14),

$$\cos(\phi - \theta_1) - \cos(\phi - \theta_2) = 0. \tag{16}$$

Hence, when the constraint is active, the relative orientation of the two vehicles must satisfy (16), and we have the following cases (see fig.1, right):

$$\text{a)} \quad \theta_1 = \theta_2; \tag{17}$$

$$\text{b)} \quad \phi - \theta_1 = \theta_2 - \phi. \tag{18}$$

In case a) the two vehicles have the same direction, while in case b) directions are symmetric with respect to the segment joining the vehicles. Constraint (13) can be rewritten as

$$\ddot{V}_{12} = 0 = 2 - 2\cos(\theta_1 - \theta_2) + \omega_1 d\sin(\theta_1 - \phi) - \omega_2 d\sin(\theta_2 - \phi), \tag{19}$$

In order to study constrained arcs of extremal solutions, jump conditions at the entry point of a constrained arc, occurring at time τ, are to be considered, and a further distinction among constrained arcs of zero and nonzero length should be done. Computations are reported in detail in [1]. For zero–length constrained arcs, jump conditions indicate precise relationships that the supporting lines of the Dubins' paths of the two vehicles, before and after the contact point, must satisfy (see for instance fig.1–left).

The case of nonzero length constrained arcs is studied below. Consider an interval $[T_1, T_2]$ during which the constraint $V_{12} \equiv 0$. As already pointed out, the study of constrained arcs of nonzero length is useful to model cooperative manipulation of object by multiple vehicles, assuming that each vehicle supports the common load through a hinge joint. A configuration of the two vehicles

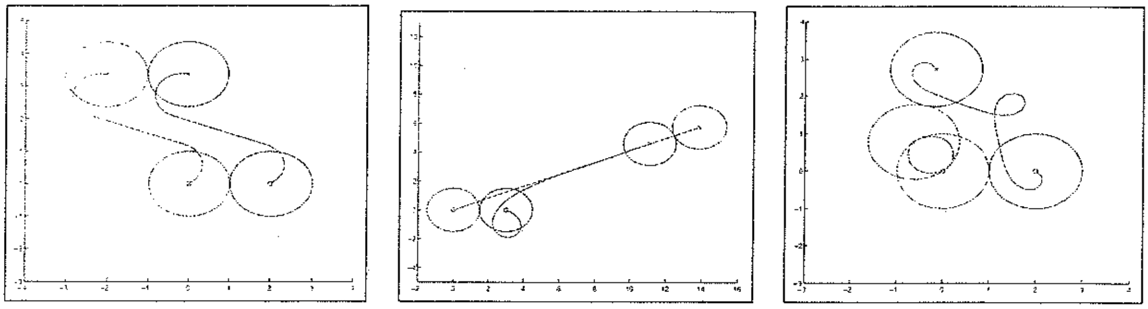

Figure 2: Left: Extremal constrained arcs of type a consist of two copies of a Dubins' path. Middle: Singular extremals in a constrained arc of type b. Right: An extremal constrained arc of type b.

along such constrained arcs can be completely described by using only four parameters, for instance the configuration (x_1, y_1, θ_1) of one vehicle and the value of ϕ. In fact, due to the tangency conditions on the constraint, one has (14) and either (17) or (18). Moreover, differentiating these relationships, one finds

$$\begin{aligned} \dot{x}_2 &= \dot{x}_1 - d\dot{\phi}\sin\phi, \\ \dot{y}_2 &= \dot{y}_1 + d\dot{\phi}\cos\phi, \end{aligned} \tag{20}$$

and

$$\dot{\phi} = \frac{1}{d}[\sin(\theta_2 - \phi) - \sin(\theta_1 - \phi)]. \tag{21}$$

Constrained arcs of nonzero length that are part of an optimal solution must themselves satisfy necessary conditions, which can be deduced by rewriting the problem in terms of the reduced set of variables. Let us consider the two types of constrained arcs separately. Notice that two extremal constrained arcs of different type may be pieced together through a configuration with $\theta_1 = \theta_2 = \phi$, which is both of type a and b.

Type a). From (21), $\phi(t) \equiv \phi_0 = \arctan \frac{y_2(0)-y_1(0)}{x_2(0)-x_1(0)}$, hence

$$\begin{aligned} \dot{x}_1 &= \dot{x}_2 \\ \dot{y}_1 &= \dot{y}_2 \\ \omega_1 &= \omega_2 \end{aligned} \tag{22}$$

Extremal constrained arcs of type a consist of a Dubins path for vehicle 1, and of a copy of the same path translated in the plane by $[d\cos\phi_0, d\sin\phi_0]^T$ for the other vehicle (see fig.2, left).

Type b). In this case, using (19), one obtains $\dot{\phi} = \frac{1}{2}(\omega_1 + \omega_2)$. From Pontryagin's minimum principle (see [1] for details) one gets that optiaml arcs of this type are either singular or nonsingular. Along a singular constrained arc of type b, one vehicle will be moving on a straight line, while the other will be trailing behind (see fig.2, middle). Nonsingular extremal constrained arcs may also obtain when a control variable is on the border of its domain, e.g. $\omega_1 = \pm\Omega_1$. In this case the motion of the two vehicles result in arcs such as those represented in fig.2, right.

2.2 Constrained arcs with $N > 2$ vehicles

The case of multiple vehicles at a fixed distance from each other allows for a multiplicity of cases (see e.g. fig.3)

Consider the case that each vehicle maintains its distance constant from two neighbors. Enumerating the vehicles sequentially, one has

$$\begin{aligned} &\cos(\phi_{j(j+1)} - \theta_{j+1}) - \cos(\phi_{j(j+1)} - \theta_j) = 0, \\ &\forall j = 1, ..., N-1 \end{aligned}$$

Centers of vehicles are on a regular N sided polyedron. Only two constrained arcs are possible in this case. Either all vehicles share the same heading angle,

$$\theta_{j+1} = \theta_j \forall j = 1, ..., N-1 \tag{23}$$

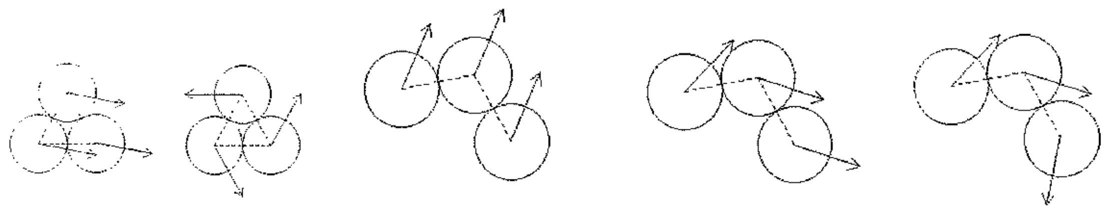

Figure 3: Different possible constrained arcs for three vehicles moving at fixed distance.

or they are moving on the circle circumscribed on the N polyedron, hence

$$\phi_{j(j+1)} - \theta_j = \frac{\pi}{N} \tag{24}$$

The circle has radius $r = \frac{d}{2*\sin(\pi/N)}$. If $r > R$, this constrained arc can only have zero length. This cases are illustrated in fig.3 (two leftmost drawings). Another possible topology for the vehicle team is the chain formation (see fig.3, three rightmnost cases). We have 2^{N-1} different types of constrained arcs in this case, for which necessary extremal conditions should be studied. Singular extremals will include at least one of the vehicles moving on a straight line. For N vehicles in chain formation, with all initial heading angles equal, optimal arcs will include Dubins' solutions.

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