Shortest Paths for Non-holonomic Vehicles with Limited Field of View Camera

Paolo Salaris*, Daniele Fontanelli*†, Lucia Pallottino* and Antonio Bicchi*

***Abstract*— This paper presents a complete characterization of shortest paths for unicycle-like nonholonomic mobile robots equipped with a limited Field-Of-View (FOV) pinhole camera rigidly fixed. We provide an alphabet of optimal control words and then we show how to obtain the shortest path, from any point of vehicle plane, that maintains a fixed point in the camera FOV.**

I. Introduction

This paper deals with the complete characterization of shortest paths for unicycle nonholonomic mobile robot equipped with a rigidly fixed pinhole camera with limited Field-Of-View (FOV). In particular, the problem considered here is to steer the mobile robot in a desired configuration through the shortest path, while keeping a specified feature in sight of a monocular, fixed camera.

The literature of optimal (shortest) paths stems mainly from the seminal work on unicycle vehicles by Dubins [9]. Dubins gave a characterization of shortest curves for a car with a bounded turning radius, proving that optimal solutions consist of combinations of circular arcs of minimum curvature (denoted by the symbols R and L for right and left arcs, respectively) and straight lines (S). He showed that an optimal path can always be found among 6 types only, described by words using at most three such symbols. A complete *optimal control synthesis* for this problem, i.e., a finite partition of the whole motion plane in regions such that the same word encodes the shortest path from all points in the same region, has been reported in [4]. Later on, the solution has been refined using results from optimal control theory [15] or Lie algebraic tools [14]. Relevant extensions have been provided in literature, including cars moving both forward and backward [11], minimum wheel rotation paths for differential-drive robots [7] or time optimal paths [16], [1].

Recently, the optimal control in the field of visual servoing has also received considerable attention, mainly for robotic manipulators (see e.g. [6]). The minimization of the manipulator trajectories between the initial and the desired positions in combination with the limited FOV constraint has been presented in [8], [10]. However, the optimal control of visually guided vehicle has received much less attention. To the best of the authors' knowledge, the work in [2], [12] represent the first attempts to find minimum length paths for nonholonomic vehicles equipped with limited FOV monocular cameras. More precisely, [2] showed that extremal arcs for the considered problem are of four types, corresponding to four symbols (rotations on the spot $*$, straight lines S and left, T^L, and right, T^R, logarithmic spirals). In [2] is stated that sequences of control symbols consist of no more than 3 symbols. In this paper, we show that shortest paths are characterized by sequences of up to 5 symbols. More details, proofs and a complete optimal synthesis of the motion plane can be found in [13].

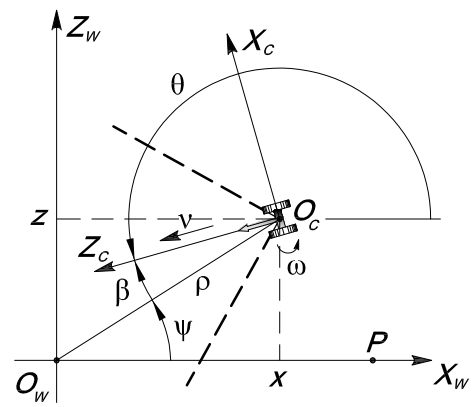

Fig. 1. Mobile robot and systems coordinates. The robot's task is to reach P while keeping O_W within a limited FOV (dashed lines).

II. Problem definition

Consider a vehicle moving on a plane where a right-handed reference frame $\langle W \rangle$ is defined with origin in O_W and axes X_w, Z_w. The configuration of the vehicle is described by $\xi(t) = (x(t), z(t), \theta(t))$, where $(x(t), z(t))$ is the position in $\langle W \rangle$ of a reference point in the vehicle, and $\theta(t)$ is the vehicle heading with respect to the X_w axis (see fig. 1). We assume that the dynamics of the vehicle are negligible, and that the forward and angular velocities of the vehicle, $v(t)$ and $\omega(t)$ respectively, are the control inputs to the kinematic model of the vehicle. Choosing polar coordinates for the vehicle (see fig. 1), i.e. setting

$$\eta = \begin{bmatrix} \rho \\ \psi \\ \beta \end{bmatrix} = \begin{bmatrix} \sqrt{x^2+z^2} \\ \arctan\left(\frac{z}{x}\right) \\ \arctan\left(\frac{z}{x}\right) - \theta + \pi \end{bmatrix}, \qquad (1)$$

The research leading to these results has received funding from the European Community's Seventh Framework Programme FP7 under grant agreement n° IST-2008-224428 "CHAT - Control of Heterogeneous Automation Systems"

* The Interdept. Research Center "Enrico Piaggio", University of Pisa, via Diotisalvi 2, 56100 Pisa, Italy. paolo.salaris,l.pallottino,bicchi@ing.unipi.it

†Department of Engineering and Information Science, University of Trento, Via Sommarive 14, 38050 Trento (TN), Italy. Phone: +390461883967. Fax:+390461882093. fontanelli@disi.unitn.it

the kinematic model of the unicycle-like robot is

$$\begin{bmatrix}\dot\rho\\ \dot\psi\\ \dot\beta\end{bmatrix}=\begin{bmatrix}-\cos\beta & 0\\ \frac{\sin\beta}{\rho} & 0\\ \frac{\sin\beta}{\rho} & -1\end{bmatrix}\begin{bmatrix}v\\ \omega\end{bmatrix}. \tag{2}$$

We consider vehicles with bounded velocities which can turn on the spot. In other words, we assume

$$(v,\omega)\in U, \tag{3}$$

with U a compact and convex subset of $\mathbb{R}^2$, containing the origin in its interior.

The vehicle is equipped with a rigidly fixed pinhole camera with a reference frame $\langle C\rangle=\{O_c,X_c,Y_c,Z_c\}$ such that the optical center O_c corresponds to the robot's center $[x(t),z(t)]^T$ and the optical axis Z_c is aligned with the robot's forward direction.

Without loss of generality, we consider the position of the robot target point P to lay on the X_W axis, with coordinates $(\rho,\psi)=(\rho_P,0)$. We also assume that the feature to be kept within the on-board camera FOV is placed on the axis through the origin O_W and perpendicular to the plane of motion.

We consider a symmetric planar FOV with characteristic angle $\delta=2\phi$, which generates the constraints

$$\beta+\phi\geq 0, \tag{4}$$
$$\beta-\phi\leq 0. \tag{5}$$

It should be noticed that we place no restrictions on the vertical dimension of the FOV. Therefore, the height of the feature on the motion plane, which corresponds to its Y_c coordinate in the camera frame $\langle C\rangle$, is irrelevant to our problem. The goal of this paper is to determine, for any point $Q\in\mathbb{R}^2$ in the robot space, the shortest path from Q to P such that the feature is maintained in the camera field of view. In other words, we want to minimize the length of the path covered by the center of the vehicle, i.e. to minimize the cost functional

$$L=\int_0^\tau |v|\,dt\,,$$

under the *feasibility constraints* (2), (3), (4), and (5). Here, τ is the time needed to reach P, i.e., such that $\rho(\tau)=\rho_P,\psi(\tau)=0$.

The time derivative of the FOV constraints computed along the trajectories of system (2) brings to

$$\dot\beta=\frac{\sin\beta}{\rho}\,v-\omega, \tag{6}$$

for both constraints. From the theory of optimal control, with state and control constraints [3], the associated Hamiltonian is

$$\begin{aligned}H(\eta,v,\omega)=&|v|-\lambda_1\cos\beta\,v+\lambda_2\frac{\sin\beta}{\rho}v+\\ &+(\lambda_3+\mu_1+\mu_2)\left(\frac{\sin\beta}{\rho}\,v-\omega\right),\end{aligned}$$

with $\lambda=(\lambda_1,\lambda_2,\lambda_3)\neq 0$ and $\mu=(\mu_1,\mu_2)\geq 0$. When the FOV constraints are not active (i.e. $\mu=0$), extremal curves (i.e. curves that satisfy necessary conditions for optimality) include straight lines (corresponding to $\omega=0$ and denoted by the symbol S) and rotations on the spot (corresponding to $v=0$ and denoted by the symbol $*$).

On the other hand, when $\mu>0$ we have

$$\begin{aligned}\beta+\phi\equiv 0 \quad&\Rightarrow\quad \tan\beta=-\tan\phi\\ \beta-\phi\equiv 0 \quad&\Rightarrow\quad \tan\beta=\tan\phi\,,\end{aligned}$$

and, by (2),

$$\dot\psi = \tan\phi\frac{\dot\rho}{\rho}=\tan\phi\frac{d}{dt}(\ln\rho)\,,\ \text{when } \beta=-\phi \tag{7}$$

$$\dot\psi = -\tan\phi\frac{\dot\rho}{\rho}=-\tan\phi\frac{d}{dt}(\ln\rho)\,,\ \text{when } \beta=\phi\,. \tag{8}$$

By integration, we obtain

$$\psi = \tan\phi\ln\left(\frac{\rho}{\rho_o}\right),\ \text{when } \beta=-\phi \tag{9}$$

$$\psi = -\tan\phi\ln\left(\frac{\rho}{\rho_o}\right),\ \text{when } \beta=\phi \tag{10}$$

where ρ_o is a constant that depends on initial conditions. Equations (9) and (10) represent two logarithmic spirals, left (denoted by the symbol T^L) and right (T^R) respectively. The left (right) spiral rotates clockwise (counterclockwise) around the feature.

We have thus obtained four extremal maneuvers, represented by the symbols $\{*,S,T^R,T^L\}$. Rotations on the spot ($*$) have zero length, but may be used to properly connect other maneuvers.

Extremal arcs can be executed by the vehicle in either forward or backward sense: we will hence use superscripts $+$ and $-$ to make this explicit (e.g., S^- stands for a straight line executed backward). In conclusion, we will build extremal paths consisting of sequences, or *words*, comprised of symbols in the alphabet $\{*,S^+,S^-,T^{R+},T^{R-},T^{L+},T^{L-}\}$. The set of possible words generated by the above symbols is a language $\mathscr{L}$.

The rest of the paper is dedicated to showing that, due to the physical and geometrical constraints of the considered problem, a sufficient optimal finite language $\mathscr{L}_O\subset\mathscr{L}$ can be built such that, for any initial condition, it contains a word describing a path to the goal which is no longer than any other feasible path.

III. Shortest paths: symmetries and invariants

For space limitations several proofs of results are omitted and only the idea behind the proof is provided. More details and all proofs can be found in [13].

By a path γ we mean a continuous map from the interval $I=[0,1]$ to the space $\mathbb{R}^2$, i.e. $\gamma:I\to\mathbb{R}^2$, with $\gamma(0)$ and $\gamma(1)$ the initial and final points of the path, respectively. We denote with $\mathscr{P}_Q$ the set of all feasible extremal paths from $\gamma(0)=Q$ to $\gamma(1)=P$.

Definition 1: Given the target point P, with $P=(\rho_P,0)$ in polar coordinates, and $Q\in\mathbb{R}^2\setminus O_W$, $Q=(\rho_Q,\psi_Q)$ with

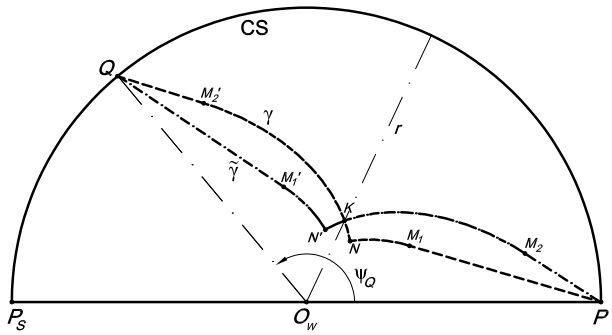

Fig. 2. Construction of a palindrome symmetric path: γ is a generic path from Q to P and $\tilde{\gamma}$ the symmetric to γ w.r.t. the bisectrix r.

$\rho_Q \neq 0$, let $f_Q : \mathbb{R}^2 \to \mathbb{R}^2$ denote the map

$$f_Q(\rho_G, \psi_G) = \begin{cases} \left(\dfrac{\rho_G \rho_P}{\rho_Q}, \psi_Q - \psi_G\right) & \text{for } \rho_G \neq 0 \\ (0,0) & \text{otherwise.} \end{cases} \tag{11}$$

Remark 1: The map f_Q can be regarded as the combination of a clockwise rotation R_Q by an angle ψ_Q, a scaling S_Q by a factor ρ_P/ρ_Q, and an axial symmetry w.r.t. X_W.

Definition 2: Given the target point $P = (\rho_P, 0)$ and $Q = (\rho_Q, \psi_Q)$ with $\rho_Q \neq 0$, let the *path transform* function F_Q be defined as

$$\begin{aligned} F_Q : \mathscr{P}_Q &\to \mathscr{P}_{f_Q(P)} \\ \gamma(t) &\mapsto f_Q(\gamma(1-t)), \forall t \in I. \end{aligned} \tag{12}$$

Remark 2: Notice that $\tilde{\gamma}(t) = F_Q(\gamma(1-t))$ corresponds to $\gamma(t)$ transformed by f_Q and reversed. Indeed, $\tilde{\gamma}$ is a path from $\tilde{\gamma}(0) = f_Q(P)$ to $\tilde{\gamma}(1) = f_Q(Q) \equiv P$.

Turning our attention back to the map $f_Q(\cdot)$, it can be noticed that point Q is transformed in $f_Q(Q) = P$, while P goes into $f_Q(P) = \left(\frac{\rho_P^2}{\rho_Q}, \psi_Q\right)$.

Consider now the locus of points Q such that it further holds $f_Q(P) = Q$. This is clearly the circumference with center in O_W and radius ρ_P. We will denote this circumference, which will have an important role in the following developments, by $C(P)$. A remarkable property of $C(P)$ is that $\forall Q \in C(P)$, $\mathscr{P}_Q$ is F_Q–invariant, i.e. $Q \in C(P) \Rightarrow \forall \gamma \in \mathscr{P}_Q$, $F_Q(\gamma) \in \mathscr{P}_Q$.

Notice that remark 1 is valid also for F_Q. As a consequence $F_{f_Q(P)}(F_Q(\gamma)) \equiv \gamma$. Furthermore, F_Q transforms forward straight lines in backward straight lines and viceversa. Moreover, F_Q maps left spiral arcs (T^{L+} and T^{L-}) in right spiral arcs (T^{R-} and T^{R+} respectively) and viceversa. Hence, F_Q maps extremal paths in $\mathscr{L}$ in extremal paths in $\mathscr{L}$. For example, let $w = S^- * T^{R-} * S^+ * T^{L+}$ be the word that characterize a path from Q to P, the transformed extremal path is of type $z = T^{R-} * S^- * T^{L+} * S^+$. With a slight abuse of notation, we will write $z = F_Q(w)$.

From the previous analysis we also obtain that an extremal path $\gamma \in \mathscr{P}_Q$ with $Q \in C(P)$ is mapped in an extremal path $\tilde{\gamma} \in \mathscr{P}_Q$ symmetric to γ w.r.t. the bisectrix r of the angle $\widehat{QO_WP}$.

In the following, we will denote by $D(P)$ the closed disc within $C(P)$. Due to the symmetry of the problem, however, the analysis of optimal paths in $\mathscr{P}_Q$ can be done considering only the upper half plane w.r.t. the X_W axis. We denote therefore by DS the closure of the semidisk in the positive Z_W half-plane, by CS the upper semicircumference, and by $\overline{P_sP}$ the diameter such that $\partial DS = CS \cup \overline{P_sP}$ – see fig. 2.

Proposition 1: Given $Q \in (X_W, Z_W)$ and a path $\gamma \in \mathscr{P}_Q$ of length l, the length of the transformed path $\tilde{\gamma} = F_Q(\gamma)$ is $\tilde{l} = \frac{\rho_P}{\rho_Q} l$.

Intuitively, the proof follows from the fact that the distance between two points P_1 and P_2 of a straight line or a logarithmic spiral linearly depends on their distance from O_W, which is scaled by means of Remark 1.

Definition 3: An extremal path starting from Q and described by a word $w \in \mathscr{L}$ is a *palindrome path* if the transformed path through F_Q is also described by w.

Definition 4: An extremal path in $\mathscr{P}_Q$ which is a palindrome path and is symmetric w.r.t. the bisectrix r of $\widehat{QO_WP}$, is said a *palindrome symmetric path*.

Proposition 2: For any path in $\mathscr{P}_Q$ with $Q \in CS$ there always exists a palindrome symmetric path in $\mathscr{P}_Q$ whose length is shorter or equal.

This can be obtained considering for any path in $\mathscr{P}_Q$ with $Q \in CS$ the transformed path by F_Q. Such two paths are symmetric with respect to the bisectrix of angle $\widehat{QO_WP}$ (see fig. 2). A palindrome symmetric path can be easily obtained considering the first part of one path and the second of the other.

Another important consequence of the properties of the path transform F_Q is the following

Theorem 1: An optimal path $\gamma \in \mathscr{P}_Q$ with $Q \in \partial DS$ evolves completely within the half-disc DS.

The proof is essentially based on Proposition 1 and the observation that a path starting outside (inside) DS is transformed in a shorter (longer) path starting inside (outside) DS.

IV. Optimal paths for points on ∂DS

Notice that the optimal path from points on $Q \in \overline{P_sO_W}$ is $S^+ * S^-$ with switching point in O_W. Trivially, for $Q \in \overline{O_WP}$ the optimal path is S^- (see fig. 2). To address the paths from points on CS, we preliminarily establish an existence result.

Proposition 3: For any $Q \in CS$ there exists a feasible shortest path to P.

Proof: Because of state constraints (4), and (5), and the restriction of optimal paths in DS (theorem 1) the state set is compact. Furthermore, for any point at distance ρ from O_W the optimal path is shorter or equal to $\rho + \rho_P$ (which corresponds to the path $S^+ * S^-$ through O_W). The system is also controllable (cf. [2]). Hence, Filippov existence theorem for Lagrange problems can be invoked [5]. ■

A sufficient family of optimal paths has to be determined. We first establish few preliminary notations and results.

Definition 5: For a point $G \in \mathbb{R}^2$, let C_G^R (C_G^L) denote the circular arc from G to O_W such that, $\forall V \in C_G^R$ (C_G^L), $\widehat{GVO_W} = \pi - \phi$ in the half-plane on the right (left) of $\overline{GO_W}$ (cf. fig. 3). Also, let C_G denote the region delimited by C_G^R and C_G^L from G to O_W.

We will refer to C_G^R (C_G^L) as the right (left) ϕ–arc in G.

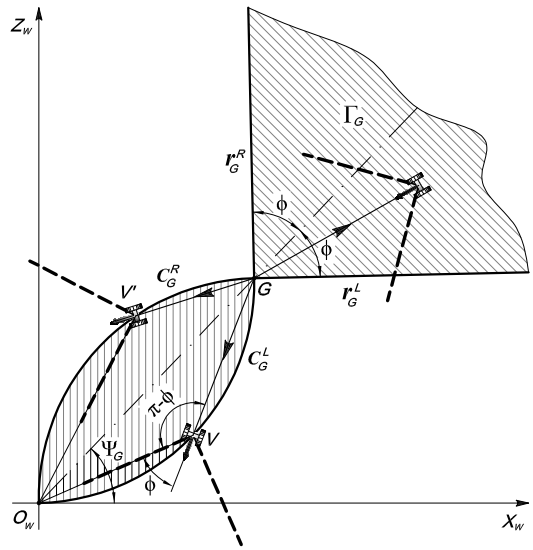

Fig. 3. Region C_Q with its border $\partial C_Q = C_Q^R \cup C_Q^L$ and cone Γ_G delimited by half–lines r_G^R and r_G^L.

Definition 6: For a point $G \in \mathbb{R}^2$, let r_G^R (r_G^L) denote the half–line from G forming an angle $\psi_G + \phi$ ($\psi_G - \phi$) with the X_W axis (cf. fig. 3). Also, let Γ_G denote the cone delimited by r_G^R and r_G^L.

We will refer to r_G^R (r_G^L) as the right (left) ϕ–radius in G. The following result is obtained by elementary geometric arguments:

Proposition 4: For any starting point Q, all points of C_Q are reachable by a straight path without violating the FOV constraint.

Proposition 5: If an optimal path $\gamma \in \mathscr{P}_Q$ includes a segment of type S^+ with extremes in A, B, then either $B = P \in C_A$ or $B \in C_A^R \cup C_A^L$.

Proposition 6: If an optimal path $\gamma \in \mathscr{P}_Q$ includes a segment of type S^- with extremes in B, A, then either $A = P \in \Gamma_B$ or $A \in r_G^R \cup r_G^L$.

Previous propositions are based on the observation that if the second extreme point is outside the region the path is not feasible. Furthermore, if it is internal the path can be shortened with a segment. Hence it is not optimal.

Proposition 7: If a path $\gamma(\tau)$, $\tau \in [0,1]$ is optimal, then its argument $\arg(\gamma(\tau))$ is monotonic.

Proof: Because γ is a continuous path, the argument of its points varies continuously. Furthermore, if the argument does not strictly decrease, than there exist two points within the path with the same argument, hence aligned with O_W. These two points could be connected with a feasible straight line, thus shortening γ, which was supposed on the contrary to be a shortest path. ■

Remark 3: By applying Proposition 7 to optimal paths from Q in the upper half–plane to P, and noticing that $\arg(Q) \geq \arg(P) = 0$, the argument is non increasing. Hence optimal paths in the upper half–plane, and in particular in DS, do not include counter–clockwise extremals of type T^{R+} or T^{L-}.

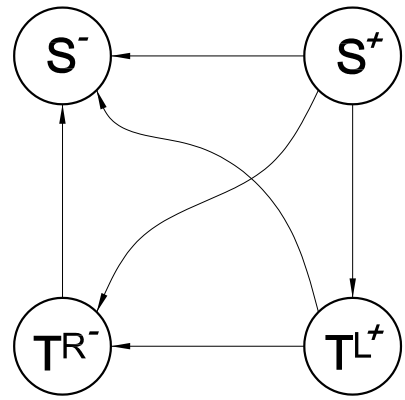

Fig. 4. Feasible extremals and sequences of extremals from points in DS.

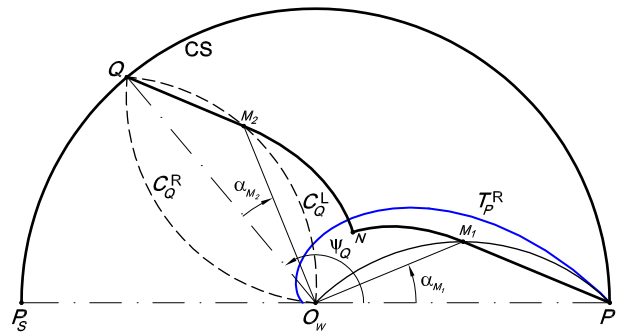

Fig. 5. The palindrome symmetric path of type $S^+T^{L+} * T^{R-}S^-$ from $Q \in CS$ to P.

Proposition 8: If a path $\gamma(\tau)$ is optimal, then its modulus $|\gamma(\tau)|$ has no local maximum for $\tau \in (0,1)$.

Proof: Because γ is a continuous path, the modulus of its points, i.e. their distance from O_W, is a continuous function of τ. Assume that the modulus has a maximum in an internal point $\bar{\tau} \in (0,1)$. Then, by classical analysis theorems, there exist two values τ_G and τ_H in $(0,1)$ such that $|\gamma(\tau_G)| = |\gamma(\tau_H)| < |\gamma(\bar{\tau})|$, with the sub–path between τ_G and τ_H evolving outside the disk of radius $|\gamma(\tau_G)|$. Applying the same arguments used in the proof of Theorem 1 replacing Q with $\gamma(\tau_G)$ and P with $\gamma(\tau_H)$, it is shown that a shorter sub–path between τ_G and τ_H exists evolving completely within the disk, a contradiction. ■

Remark 4: Observe that the distance from O_W is strictly increasing along backward extremal arcs (i.e. S^-, T^{R-}, T^{L-}) and strictly decreasing along forward extremal arcs (i.e. S^+, T^{R+}, T^{L+}). As a consequence of Proposition 8 in an optimal path a forward arc cannot follow a backward arc.

Proposition 9: Any path of type $S^- * T^{R-}$ ($T^{L+} * S^+$) can be shortened by a path of type $T^{R-}S^-$ (S^+T^{L+}).

The proof of this proposition can be found in [13].

We are now able to prove the following important result.

Theorem 2: From each and every $Q \in CS$ to P there exists a palindrome symmetric shortest path of type $S^+T^{L+} * T^{R-}S^-$.

Proof: According to Propositions 7–9 and remarks 3–4, a sufficient optimal language $\mathscr{L}_O$ for $Q \in DS$ is described in fig. 4. It is straightforward to observe that the number

of switches between extremals is finite and less or equal to 3, and a sufficient family of optimal paths is given by the word $S^+T^{L+} * T^{R-}S^-$ and its degenerate cases. Furthermore, by Proposition 2, for $Q \in CS$ optimal paths are palindrome symmetric paths. ■

A palindrome symmetric path from Q on CS to P of the type $S^+T^{L+} * T^{R-}S^-$ is shown in fig. 5. By symmetry, it follows that the sub–paths S^+ and S^- have the same length, and so do T^{L+} and T^{R-}. As a consequence, only two sub-words $T^{L+} * T^{R-}$ and $S^+ * S^-$ need be considered, which are obtained as degenerate cases with zero length arcs.

Referring to fig. 5, let the switching points of the optimal path be denoted as M_2, N, and M_1, respectively. Notice that N is on the bisectrix r of $\widehat{QO_WP}$, while M_1 and M_2 are symmetric w.r.t. r. In fig. 5 the region C_Q, locus of points reachable by a linear feasible path from Q, is also reported delimited by dashed curves.

We now study the length of extremal paths from CS to P in the sufficient family above. To do so, it is instrumental to parameterize the family by the angular position of the first switching point, α_{M_1}.

Theorem 3: The length of a path $\gamma \in \mathscr{P}_Q$, $Q \in CS$, of type $S^+T^{L+} * T^{R-}S^-$ passing through $M_1 = (\rho_{M_1}, \alpha_{M_1})$ is

$$L = 2\frac{\rho_P}{\cos\phi}\cos\alpha_{M_1} - \frac{2\rho_P e^{\left(\alpha_{M_1} - \frac{\psi_Q}{2}\right)t}}{\cos\phi\sin\phi}\sin(\phi - \alpha_{M_1}). \quad (13)$$

The result has been obtained considering that a shortest path is palindrome symmetric (Proposition 2). Hence, $M_1 \in C_P^R$, $M_2 \in C_Q^L$, $\rho_{M_2} = \rho_{M_1}$ and $\alpha_{M_2} = \psi_Q - \alpha_{M_1}$.

Having an analytical expression for the length of the path as a function of a single parameter α_{M_1} (hence indirectly of $Q \in CS$), we are now in a position to minimize the length within the sufficient family. Notice that we need only to consider $\alpha_{M_1} \geq 0$ (because the problem is symmetric w.r.t. X_W), and $\alpha_{M_1} \leq \phi$ for the geometrical considerations above on C_Q^L (see fig. 3).

Theorem 4: Given $Q = (\rho_P, \psi_Q) \in CS$,

- for $0 < \psi_Q \leq \psi_M := -4\tan\phi\ln(\sin\phi)$, the optimal path is of type $T^{L+} * T^{R-}$;
- for $\psi_M < \psi_Q < \psi_V := 2\phi + \psi_M$, the optimal path is of type $S^+T^{L+} * T^{R-}S^-$;
- for $\psi_V \leq \psi_Q < \pi$, the optimal path is of type $S^+ * S^-$

With reference to fig. 6, the locus of switching points between extremals in optimal paths is defined in the following proposition.

Proposition 10: Given $Q = (\rho_P, \psi_Q) \in CS$,

- for $0 < \psi_Q \leq \psi_M$, the switching locus is the arc of T_P^R within the extreme points P and $m = (\rho_P \sin^2\phi, \psi_M/2)$ (included);
- for $\psi_M < \psi_Q < \psi_V$, the loci of switching points M_1, N, and M_2 are the right ϕ–arcs C_P^R, C_m^R, and C_M^R with $M = (\rho_P, \psi_M)$, respectively;
- for $\psi_V \leq \psi_Q < \pi$, the switching locus reduces to the origin O_W.

We are now able to provide an explicit procedure to compute the switching points for any given $Q \in CS$:

- for $0 < \psi_Q \leq \psi_M$, the switching point is $T_P^R \cap T_Q^L$;

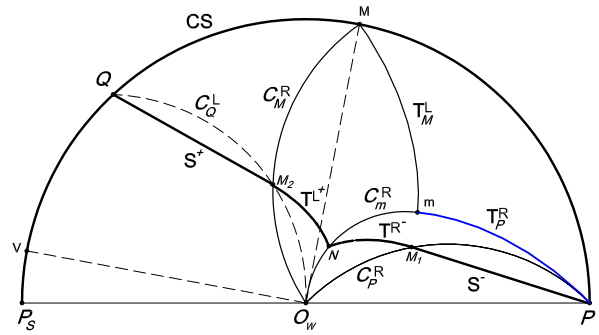

Fig. 6. Optimal path from Q on CS, between M and V, to P. The locus of switching points between extremals S^+ and T^{L+} is the arc of circle C_M^R, whereas the locus of switching points between T^{L+} and T^{R-} is C_m^R.

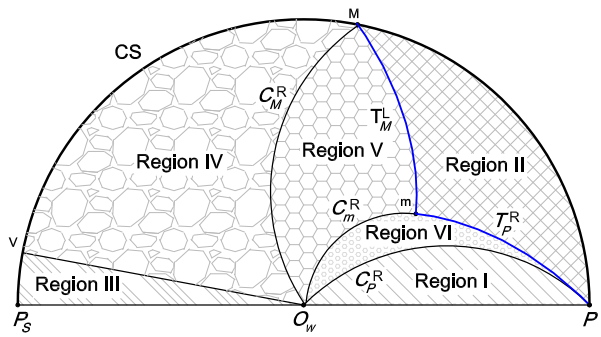

Fig. 7. Partition of DS.

- for $\psi_M < \psi_Q < \psi_V$, the switching points are $M_2 \in C_M^R \cap C_Q^L$, $N \in C_m^R \cap T_{M_2}^L$, and $M_1 \in C_P^R \cap T_N^R$.
- for $\psi_V \leq \psi_Q < \pi$, the switching point is O_W.

Proofs of previous results can be found in [13].

V. Shortest paths from any point in the motion plane

For any points on CS the word that characterize the shortest path has been found (Theorem 4) and a way to determine the switching points has been provided (Prop 10). We are now able to determine the shortest path for internal points in DS by using the following simple idea: for any $Q \in DS \setminus \partial DS$, find a point s (e.g. $s \in \partial DS$) such that an, already known, optimal path γ from s to P goes through Q. By Bellmann's optimality principle, the sub–path from Q to P is also optimal. Once shortest paths for points of DS are derived, the optimal paths from points in the rest of the plane are obtained using properties of map F_Q.

A. Optimal paths for points in the half-disc DS

Referring to fig. 7 switching locus of optimal paths from CS subdived DS in 6 Regions. Notice that a shortest path from CS cross Region I with the S^- type of extremal, Regions II and V with T^{L+}, Regions III and IV with S^+ and Region VI with T^{R-}.

For any point Q in DS a shortest path can be obtained as follows determining a point on CS from which the shortest path is through Q.

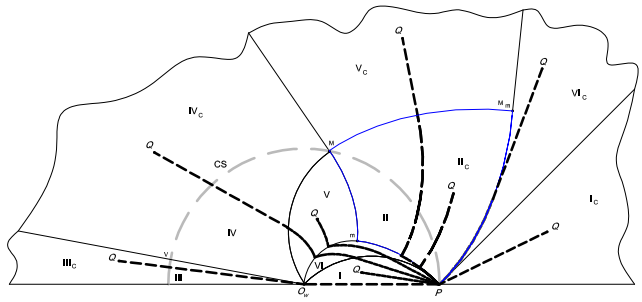

Fig. 8. Examples of optimal paths from points Q in different regions to P.

Region I: From any point in this region it is possible to reach P with a straight path (in backward motion) without violating the FOV constraints (cf. Proposition 4).

Region II: For any Q in Region II consider the point s obtained by intersecting the spiral T_Q^L with CS. By the non-intersecting properties of left spirals, s lies between P and M on CS.

Region II$'$: For any Q in the arc of T_P^R from m to P, the path T_P^{R-} from Q to P is a degenerate case of $T^{L+} * T_P^{R-}$ with a zero-length T^{L+} arc.

Region III: For any Q in Region III consider the line through O_W and Q, which intersects CS in a point s between V and P_s.

Region IV: For any Q in Region IV consider the left ϕ–arc C_Q^L, and the intersection point $r = C_Q^L \cap C_M^R \setminus O_W$.

Consider now the straight line through Q and r, and let its intersection with CS be denoted s. Such intersection lies between V and M.

We recall also that, as a straightforward consequence of Proposition 10, the optimal path for $Q \in C_m^R$ is of type $T_Q^{R-}S^-$, while for $Q \in C_M^R$, the optimal path type is $T_Q^{L+} * T^{R-}S^-$, where the two spiral extremals have the same length. Hence we have:

Region V: For any Q in Region V consider the intersection point s of the spiral T_Q^L with C_M^R. The optimal path γ_s from $s \in C_M^R$ to P is of type $T_s^{L+} * T^{R-}S^-$, and contains Q in its first arc.

Region VI: For any Q in Region VI consider the intersection point s of the spiral T_Q^R with C_m^R. The optimal path γ_s from $s \in C_m^R$ to P is of type $T_s^{R-}S^-$ and contains Q in its first arc.

B. Optimal paths for points outside DS

To obtain shortest paths for points outside DS we recall that (section III) F_Q transforms a path from Q to P in a path from $f_Q(P) = \left(\frac{\rho_P^2}{\rho_Q}, \psi_Q\right)$ to P. Recall also that, if Q is outside the half–disk DS, $f_Q(P)$ is inside, and viceversa.

From proposition 1 we obtain that optimal paths from points Q outside DS can be obtained considering the shortest path γ from $\tilde{Q} = f_Q(P)$ and mapping γ by $F_{\tilde{Q}}$.

Examples of shortest paths from different region are depicted in fig. 8.

VI. Conclusions and Future Work

A complete characterization of shortest paths for unicycle nonholonomic mobile robots equipped with a rigidly fixed pinhole camera with limited FOV has been proposed. In particular, new optimal words has been considered with respect to previous works and optimal paths from any point of the motion plane have been obtained for any value of the FOV characteristic angle.

A possible extension of this work is to consider also a vertical limit of the FOV that induces an off-limit zone close to the feature position. Furthermore, the problem of determining optimal paths in the case of more than one feature can also be considered.

References

[1] D. Balkcom and M. Mason, "Time-optimal trajectories for an omni-directional vehicle," *The International Journal of Robotics Research*, vol. 25, no. 10, pp. 985–999, 2006.

[2] S. Bhattacharya, R. Murrieta-Cid, and S. Hutchinson, "Optimal paths for landmark–based navigation by differential–drive vehicles with field–of–view constraints," *IEEE Transactions on Robotics*, vol. 23, no. 1, pp. 47–59, February 2007.

[3] A. Bryson and Y. Ho, *Applied optimal control*. Wiley New York, 1975.

[4] X. Bui, P. Souères, J.-D. Boissonnat, and J.-P. Laumond, "Shortest path synthesis for Dubins non–holonomic robots," in *IEEE International Conference on Robotics and Automation*, 1994, pp. 2–7.

[5] L. Cesari, *Optimization-theory and applications: problems with ordinary differential equations*. Springer-Verlag, New York, 1983.

[6] G. Chesi and Y. Hung, "Global path-planning for constrained and optimal visual servoing," *IEEE Transactions on Robotics*, vol. 23, no. 5, pp. 1050–1060, October 2007.

[7] H. Chitsaz, S. M. LaValle, D. J. Balkcom, and M. Mason, "Minimum wheel-rotation for differential-drive mobile robots," *The International Journal of Robotics Research*, pp. 66–80, 2009.

[8] K. Deguchi, "Optimal motion control for image-based visual servoing by decoupling translation and rotation," in *IEEE/RJS Intl. Conf. on Intelligent Robots and Systems*, Victoria, B.C., Canada, October 1998, pp. 705–711.

[9] L. E. Dubins, "On curves of minimal length with a constraint on average curvature, and with prescribed initial and terminal positions and tangents," *American Journal of Mathematics*, pp. 457–516, 1957.

[10] Y. Mezouar, A. Remazeilles, P. Gros, and F. Chaumette, "Images interpolation for image-based control under large displacement," in *IEEE Intl. Conf. on Robotics and Automation*, Washington, DC, May 2002, pp. 3787–3794.

[11] J. A. Reeds and L. A. Shepp, "Optimal paths for a car that goes both forwards and backwards," *Pacific Journal of Mathematics*, pp. 367–393, 1990.

[12] P. Salaris, F. A. W. Belo, D. Fontanelli, L. Greco, and A. Bicchi, "Optimal paths in a constrained image plane for purely image-based parking," *IROS*, pp. 1673–1680, 2008.

[13] P. Salaris, D. Fontanelli, L. Pallottino, and A. Bicchi, "Shortest paths for a robot with nonholonomic and field-of-view constraints," *IEEE Transactions on Robotics*, 2009.

[14] H. Souères and J. P. Laumond, "Shortest paths synthesis for a car-like robot," *IEEE Transaction on Automatic Control*, pp. 672–688, 1996.

[15] H. Sussmann and G. Tang, "Shortest paths for the reeds-shepp car: A worked out example of the use of geometric techniques in nonlinear optimal controlò," Department of Mathematics, Rutgers University, Tech. Rep., 1991.

[16] H. Wang, Y. Chan, and P. Souères, "A geometric algorithm to compute time-optimal trajectories for a bidirectional steered robot," *IEEE Transaction on Robotics*, pp. –, 2009.