LEFT INVERTIBILITY OF DISCRETE SYSTEMS WITH FINITE INPUTS AND QUANTIZED OUTPUT

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ABSTRACT. The aim of this paper is to address left invertibility for dynamical systems with inputs and outputs in discrete sets. We study systems which evolve in discrete time within a continuous state-space; quantized outputs are generated by the system according to a given partition of the state-space, while inputs are arbitrary sequences of symbols in a finite alphabet, which are associated to specific actions on the system. Our main results are obtained under some contractivity hypotheses. The problem of left invertibility, i.e. recovering an unknown input sequence from the knowledge of the corresponding output string, is addressed using the theory of Iterated Function Systems (IFS), a tool developed for the study of fractals. We show how the IFS naturally associated to a system and the geometric properties of its attractor are linked to the invertibility property of the system. Our main result is a necessary and sufficient condition for left invertibility and uniform left invertibility for joint contractive systems. In addition, an algorithm is proposed to recover inputs from output strings. A few examples are presented to illustrate the application of the proposed method.

1. INTRODUCTION

Invertibility of dynamical systems is a fundamental problem of systems theory, and is distinguished in two aspects: right invertibility, which is concerned with surjectivity of the I/O map; and left invertibility, corresponding to injectivity of the map. While right inversion allows to find inputs and initial conditions which can produce a given output, left invertibility deals with the possibility of recovering unknown inputs applied to the system from the knowledge of the outputs.

Invertibility problems are of interest in applications like fault detection in Supervisory Control and Data Acquisition (SCADA) systems, system identification, and cryptography [11, 18]. Invertibility of linear systems is a well understood problem, pioneered by [5], and then considered with algebraic approaches [30], frequency domain techniques [20, 21], and geometric tools [22]. More recent work has addressed the invertibility of nonlinear systems [27]. Right-invertibility is studied with differential geometry methods for instance in [23] and [26] for classes of smooth nonlinear systems. In [34], the left invertibility problem for a switched system is discussed.

This paper deals with left invertibility of a class of discrete–time nonlinear dynamical systems in a continuous state-space with inputs and outputs in discrete sets. In particular, we consider the case in which inputs are arbitrary sequences of symbols in a finite alphabet, each symbol being associated to a specific action on the system. Information available on the system is represented by sequences of output values in a discrete set. Such outputs are obtained by quantization, i.e. are generated by the system evolution according to a given partition of the state-space.

Quantized control systems have been attracting increasing attention of the control community in recent years (see for instance [9, 24, 25] and references therein). The mathematical operation of quantization and the possibility of considering only finite inputs occurs in many communication and control systems [19, 32]. Finite inputs may arise because of

the intrinsic nature of the actuator, or anyway wherever the system operates under a logical supervisor. On the other hand, output quantization may occur because of the digital nature of the sensor, or if data need to be digitally transmitted. Most recently, the attention to quantization has been stimulated by the growing number of application involving "networked" control systems, which are systems interconnected through channels of limited capacity [2, 7, 33].

The problem considered in this paper is that of determining whether a given quantized system is left invertible (LI). To this purpose, we first define the properties of distinguishability and uniform distinguishability of two input sequences. Loosely speaking, two input sequences are distinguishable if they generate two output strings that differ from each other on a finite time horizon. The main tool used in the paper is the theory of Iterated Function Systems (IFS), developed for the study of fractals. IFS have been already used as a model in different fields [6, 28]. One can construct a natural map in the space of compact subsets of the euclidean space, simply by mapping a set in the union of the images of all maps forming the IFS. Under some contraction hypotheses the resulting dynamical system has a unique attractor, which is also a compact set. Using recent results, we can determine the properties of the original control systems in terms of such compact attractor. More precisely, for every finite subset C of the finite input alphabet, there exists an attractor $\mathscr{A}_C$. If all attractors are not inside a particular "diagonal" set, then almost every couple of distinct output strings is distinguishable (see Theorem 2). These results are valid in a probabilistic sense, i.e. they hold with probability one with respect to the invariant probability measure for the given IFS.

The property of uniform left invertibility is of even greater interest for applications. We address such problem using a graph that is associated to the attractor: paths on this graph are associated to orbits of the system. The main result about uniform left invertibility is Theorem 3, that provides necessary and sufficient conditions for left invertibility. The Random Iteration Algorithm, abbreviated with RIA, [10] is also useful to study uniform left invertibility. This consists simply in randomly choosing input sequences linked to given probability distribution functions, and generating the corresponding orbits of the system. A recent result in dynamical system theory (Theorem 7) indicates that the asymptotic probability of belonging to a given set in the state space is equal to the measure of the set, for the probability measure which is invariant for the IFS. Moreover such number can be computed by the RIA as its limiting behavior. Thus, a strategy can be set up using the RIA to estimate such limit and then to derive information about the average size of time intervals during which an orbit remains inside the given set. This provides in turn an average of the waiting time for uniform left invertibility, with probability one. Finally, we illustrate our approach on examples.

The paper is organized as follows. In Section 2 we review the background concepts. Section 3 defines simple, uniform, and almost everywhere (AE) distinguishability and left invertibility, and shows the link between IFS theory and invertibility: our main result, Theorem 3, gives necessary and sufficient conditions for left invertibility and uniform left invertibility. In section 4 a computationally tractable test probabilistic for AE left invertibility is proposed, exploiting results about the invariant measure associated to an IFS. Section 5 contains an algorithm to detect inputs, under the assumption of uniform left invertibility. In section 6 we present examples about the application of the method described in sections 3 and 4. Section 7 shows conclusions and future perspectives.

2. BASIC SETTING AND BACKGROUND

In this paper we consider discrete-time, autonomous, non-linear systems of the form

$$\begin{cases} x(k+1) = f\big(x(k), u(k)\big) = f_{u(k)}\big(x(k)\big) \\ y(k) = q\big(x(k)\big) \end{cases} \tag{1}$$

where $x(k) \in \mathbb{R}^d$ is the state, $y(k) \in \mathscr{Y}$ is the output, and $u(k) \in \mathscr{U}$ is the input. We assume that $\mathscr{Y}$ is a discrete set. The map $q : \mathbb{R}^d \to \mathscr{Y}$ is induced by a locally finite partition $\mathscr{P} = \dot\bigcup_{i \in \mathscr{Y}} \mathscr{P}_i$ of $\mathbb{R}^d$ ($\dot\bigcup$ denotes the disjoint union) through $q : (x \in P_i) \mapsto i$ and will be referred to as the output quantizer. We admit infinite partitions, but we assume that $\mathscr{U}$ is a finite set of cardinality m. With no loss of generality (modulo redefining the dynamics $f(\cdot, \cdot)$ and the function q), we will assume $\mathscr{U} = \{1, \ldots, m\}$.

Remark 1. *The results in this paper are indeed valid in every complete metric space: the properties we are using depend only on the metric. We have chosen $\mathbb{R}^d$ to be the state space for the sake of simplicity and to avoid some technicalities.* ◇

Remark 2. *If a real number p_i with $0 < p_i < 1$, $\sum_{i=1}^m p_i = 1$, is associated to each symbol $u_i \in \{1, \ldots, m\}$, this can be interpreted as the probability that the symbol appears, i.e. the event $\{u(k) = u_i\}$ occurs. This association entails a probabilistic interpretation of the results in this paper. However, as we will observe explicitly, the main results about left invertibility and uniform left invertibility of a system (Theorems 2, 3) do not depend on the particular choice of the p_i's.* ◇

Example 1. *We call a system of type (1) I/O quantized linear if it is of the form*

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) \\ y(k) = q\big(Cx(k)\big) \end{cases} \tag{2}$$

where $x(k) \in \mathbb{R}^d$, $y(k) \in \mathbb{Z}^p$, $A \in \mathbb{R}^{d \times d}$, $B \in \mathbb{R}^{d \times m}$, $u(k) \in \mathscr{U} \subset \mathbb{R}^m$ is the input, $C \in \mathbb{R}^{p \times d}$, and $q : \mathbb{R}^d \to \mathbb{Z}$ is any quantizer. ◇

If x_0 is an initial condition, $k_1 < k_2 \in \mathbb{N}$ and $(u_1, \ldots, u_{k_2})$ a sequence of inputs, we let $f_{k_1}^{k_2}(x_0, u_1, \ldots, u_{k_2})$ denote the sequence of outputs $(y_{k_1}, \ldots, y_{k_2})$ generated by the system (1) with initial condition x_0 and input string $(u_1, \ldots, u_{k_2})$.

2.1. **Contractive IFS theory.** In this paragraph we collect some basic results from the Iterated Function System theory.

Definition 1. *Let $(\mathbb{X}, d)$ be a complete metric space. A map $F : \mathbb{X} \to \mathbb{X}$ is contractive if $\exists c \in \mathbb{R}$, $0 < c < 1$ such that $d\big(F(x), F(y)\big) \leq cd(x, y)$ for all $x, y \in \mathbb{X}$.*
A map $F : \mathbb{X} \to \mathbb{X}$ is expansive if $\exists e > 1$ such that $d\big(F(x), F(y)\big) \geq ed(x, y)$ for all $x, y \in \mathbb{X}$. ◇

Example 2. *A linear map is contractive if its associated matrix has norm less than 1, where the norm of a matrix A is defined by $\|A\| = \sup_{x \in \mathbb{R}^d} \frac{\|Ax\|}{\|x\|}$. A linear map is expansive if for its associated matrix holds $\inf_{x \in \mathbb{R}^d} \frac{\|Ax\|}{\|x\|} > 1$.* ◇

Definition 2. *An Iterated Function System with probabilities is a collection*

$$\{\mathbb{X}, F_1, \ldots, F_n, p_1, \ldots, p_n\}, \tag{3}$$

where $(\mathbb{X}, d)$ is a metric space, $F_i : \mathbb{X} \to \mathbb{X}$ for $i = 1, \ldots, n$, and $p_i \in \mathbb{R}$ such that $\sum_{i=0}^n p_i = 1$, $0 < p_i < 1$, for $i = 1, \ldots, n$. When the p_i's are not specified we refer to $\{\mathbb{X}, F_1, \ldots, F_n\}$ simply as an IFS. ◇

An Iterated Function System with probabilities is related to a Markov jump system. Investigations about Markov jump systems include for instance almost sure stability [4] and methods of secure communication [29]. In this paper we use the theory of Iterated Function Systems to attack left invertibility of such systems.

Definition 3. *Given an IFS $\{\mathbb{X}, F_1, \ldots, F_n\}$ such that F_i is contractive for every i, define the contractivity factor of the IFS to be*

$$s = \min \big\{ c \in \mathbb{R} : \ \forall i = 1, \ldots, n \ \forall x, y \in \mathbb{X} \ d(F_i(x), F_i(y)) \leq cd(x, y) \big\}.$$ ◇

Definition 4. *Given the IFS with probability (3), define*

$$\mathfrak{F}_p = \left\{ \{F_{i_1} \circ \ldots \circ F_{i_p}\} : i_1, \ldots, i_p \in \{1, \ldots, n\} \right\}.$$

The IFS is joint contractive if there exists $p \in \mathbb{N}$ such that all elements of $\mathfrak{F}_p$ are contractions. The IFS is joint expansive if there exists $p \in \mathbb{N}$ such that all elements of $\mathfrak{F}_p$ are expansive. ◇

Example 3. *An I/O quantized linear system is joint contractive if and only if for every eigenvalue λ of the matrix A it holds $|\lambda| < 1$. It is joint expansive if and only for every eigenvalue λ of the matrix A it holds $|\lambda| > 1$.* ◇

We refer to [1, 13] for general theory of Iterated Function Systems (also called Iterated Function Schemes). In what follows we use $\sigma = \{\sigma_i\}_{i=1}^{\infty}$ to indicate a sequence of indices in $\{1, \ldots, n\}$. Moreover, for every $C \subset \{1, \ldots, n\}$ we indicate by Σ_C the set of all sequence in C, and we let $\Sigma = \Sigma_{\{1,\ldots,n\}}$.

Definition 5. *An orbit for the IFS (3) is a sequence $\{x(k)\}_{k=0}^{\infty} = \{x(k)_{x(0),\sigma}\}_{k=0}^{\infty} \subset \mathbb{X}$ given by the choice of an initial condition $x(0) \in \mathbb{X}$ and a sequence $\sigma \in \Sigma$, according to the following rule: $x(k+1) = F_{\sigma_k}(x(k))$.* ◇

We now define, in a standard way, a measure on $\{1, \ldots, n\}^{\mathbb{N}}$.

Definition 6. *For $i_1, \ldots, i_r \in \mathbb{N}$, $j_1, \ldots, j_r \in C$, the cylindrical subsets $v_{i_1,\ldots,i_r}^{j_1,\ldots,j_r}$ of Σ_C, is the set of strings defined by:*

$$\sigma \in v_{i_1,\ldots,i_r}^{j_1,\ldots,j_r} \quad \Leftrightarrow \quad \sigma_k = \begin{cases} j_1 & for\ k = i_1 \\ \ldots \\ j_r & for\ k = i_r \end{cases} \quad ◇$$

A cylindrical subset $v_{i_1,\ldots,i_r}^{j_1,\ldots,j_r}$ is the set of all strings for which the $i_k - th$ component assumes the value j_k, for $k = 1, \ldots, r$. The collection of all cylindrical subsets of Σ generates a $\sigma-$algebra $\mathscr{B}$ on Σ. On these subsets we define the measure μ by

$$\mu[v_{i_1,\ldots,i_r}^{j_1,\ldots,j_r}] = p_{j_1} \cdot \ldots \cdot p_{j_r}. \tag{4}$$

This corresponds to the fact that the probability of the choice of the map F_i is p_i independently of the time. Equality (4) uniquely defines a probability measure on the entire $\sigma-$algebra $\mathscr{B}$ denoted by the same symbol μ [16].

Definition 7. *[3] A set $\mathscr{A}_C \subset \mathbb{X}$ is an attractor for the IFS (3) with respect to the index set C if for all initial condition $x(0) \in \mathbb{X}$ and for all $\sigma \in \Sigma_C$ $\lim_{k\to\infty} d(x(k)_{x(0),\sigma}, \mathscr{A}_C) = 0$, where $d(x, \mathscr{A}_C) = \inf_{a \in \mathscr{A}_C} d(x, a)$.* ◇

The orbit is thus forced to asymptotically approach the attractor.

Definition 8. *A set $\mathscr{I}_C \subset \mathbb{X}$ is an invariant set for the IFS (3) with respect to the index set C if*

$$\mathscr{I}_C = \bigcup_{i \in C} F_i(\mathscr{I}_C). \ ◇ \tag{5}$$

Note that, if $\mathscr{I}_C$ is an invariant set, given any initial condition $x(0)$ in $\mathscr{I}_C$, every possible orbit of the IFS (3) with indices in C is contained in $\mathscr{I}_C$. The next two results show that attractors and invariant sets exist for joint contractive IFS and are compact for all input sets C.

Theorem 1. *Let the IFS (3) be joint contractive and let $C \subset \{1, \ldots, n\}$ be given. Then, for every $\sigma \in \Sigma_C$ the limit*

$$\phi(\sigma) = \lim_{k \to \infty} F_{\sigma_1} \circ \ldots \circ F_{\sigma_k}(x)$$

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exists for every $x \in \mathbb{X}$ and is independent of x. The set $\phi(\Sigma_C) = \mathscr{A}_C$ is the unique compact attractor and invariant set with respect to $\{F_i : i \in C\}$. Moreover, for all initial condition $x(0)$ and $\mu - AE$ $\sigma \in \Sigma_C$, the set $\{x(k)\}_{k \in \mathbb{N}} \bigcap \mathscr{A}_C$ is dense in $\mathscr{A}_C$.

Proof: See [15] and [3]. ◇

Proposition 1. [1] *Let $C_1, C_2 \subseteq \{1, \ldots, n\}$ be such that $C_1 \subseteq C_2$. Then $\mathscr{A}_{C_1} \subseteq \mathscr{A}_{C_2}$* ◇.

The attractor $\mathscr{A}_C$ in Theorem 1 is easily algorithmically computable with the so called Random Iteration Algorithm [10]. See section 4 for further details.

Definition 9. *An address of a point $a \in \mathscr{A}_{\{1,\ldots,n\}}$ is any member of the set $\phi^{-1}(a) = \{\sigma \in \Sigma : \phi(\sigma) = a\}$. The attractor is said to be totally disconnected if each point possesses a unique address.* ◇

Proposition 2. [1] *The attractor $\mathscr{A} = \mathscr{A}_{\{1,\ldots,n\}}$ is totally disconnected if and only if $F_i(\mathscr{A}) \cap F_j(\mathscr{A}) = \emptyset$ $\forall i \neq j$.* ◇

3. Attractors and left invertibility

In this section we define left invertibility of a system, and we show that, if the system is joint contractive, left invertibility is strictly connected with the properties of the system's attractor.

Definition 10. *A pair of inputs strings $u = \{u_i\}_{i \in \mathbb{N}}$, $u' = \{u'_i\}_{i \in \mathbb{N}}$ is distinguishable if $\forall x_0, x'_0 \in \mathbb{R}^d$ $\exists k = k(x_0, x'_0, \{u_i\}, \{u'_i\}) \in \mathbb{N}$ such that*

$$f_0^k(x_0, u_1, \ldots, u_k) \neq f_0^k(x'_0, u'_1, \ldots, u'_k). \diamond$$

Definition 11. *A pair of input strings $\{u_i\}_{i \in \mathbb{N}}$, $\{u'_i\}_{i \in \mathbb{N}}$ is uniformly distinguishable in k steps, $k \in \mathbb{N}$, (or with distinguishability time k) if for every compact set $K \in \mathbb{R}^d \times \mathbb{R}^d$ there exists $l = l(K)$ such that $\forall (x_0, x'_0) \in K$ and $\forall m > l$ the following holds:*

$$u_m \neq u'_m \Rightarrow f_m^{m+k}(x_0, u_1, \ldots, u_{m+k}) \neq f_m^{m+k}(x'_0, u'_1, \ldots, u'_{m+k}).$$

In this case, we say that the strings are uniformly distinguishable with waiting time l. ◇

Fact: A pair of strings is not uniformly distinguishable if and only if there exists a compact set $K \in \mathbb{R}^d \times \mathbb{R}^d$ with the following property: $\forall k \in \mathbb{N}$ $\exists (x_0, x'_0) \in K$ such that

$$u_m \neq u'_m \quad and \quad f_m^{m+k}(x_0, u_1, \ldots, u_{m+k}) = f_m^{m+k}(x'_0, u'_1, \ldots, u'_{m+k}) \tag{6}$$

for some $m \in \mathbb{N}$.

Proof: Suppose that two strings are not uniformly distinguishable. Then, there there exists a compact set $K \in \mathbb{R}^d \times \mathbb{R}^d$ with the property: $(\forall l)$ $\forall k \in \mathbb{N}$ $\exists (x_0, x'_0) \in K$ such that (6) holds. So the necessity proof is complete.
For the sufficiency, if $\forall k \in \mathbb{N}$ $\exists (x_0, x'_0) \in K$ such that (6) holds for some $m \in \mathbb{N}$, then the existence of a waiting time (*l* in the definition of uniform distinguishability) is not possible. ◇

Definition 12. *A system is left invertible (LI) if for any two input strings u, u' there exists $l(u, u') \in \mathbb{N}$ such that, if $u_i \neq u'_i$, $i > l$, the outputs after the instant l are different for any pair of initial states, i.e. u, u' are distinguishable.* ◇

So, for a LI system, it is possible to recover *infinite* input strings observing the corresponding *infinite* output strings.

Definition 13. *A system of type (1) is uniformly left invertible (ULI) in k steps if, for initial conditions in a compact set $K \subset \mathbb{R}^d \times \mathbb{R}^d$, every pair of distinct input sequences is uniformly distinguishable in k steps after a finite time l, where k is constant and l depends only on K.* ◇

For a ULI system, it is possible to recover the input string until instant m observing the output string until instant $m + k$. For applications, however it is important to obtain an algorithm to reconstruct the input symbol used at time $m > l$ by processing the output symbols from time m to $m + k$.

Definition 14. *A system of type (1) is μ-AE LI if for μ-almost every couple of strings u, u' there exists $l(u, u') \in \mathbb{N}$ such that, if $u_i \neq u'_i$, $i > l$, the outputs after the instant l are different for any pair of initial states.*
A system of type (1) is μ-AE uniformly left invertible in k steps if, for initial conditions in a compact set $K \subset \mathbb{R}^d \times \mathbb{R}^d$, μ-almost every pair of distinct input sequences are uniformly distinguishable in k steps after a finite time l, where k is constant and l depends only on K (μ is the measure defined in (4)). ◇

We introduce now a technique that links the left invertibility problem to the theory of Iterated Function Systems and the properties of their attractors, so that we can apply all the results described in Section 2. Define

$$Q = \bigcup_{y \in \mathscr{Y}} \{q^{-1}(y) \times q^{-1}(y)\} \subset \mathbb{R}^{2d}$$

i.e. the union of the preimages of two identical output symbols. In other words, Q contains all pairs of states that are in the same element of the partition $\mathscr{P}$. To address left invertibility, we are interested in studying the following system on $\mathbb{R}^{2d}$:

$$X(k+1) = F_{U(k)}(X(k)) = \begin{bmatrix} f(x_1(k), u(k)) \\ f(x_2(k), u'(k)) \end{bmatrix} \tag{7}$$

where $X(k) = \begin{pmatrix} x_1(k) \\ x_2(k) \end{pmatrix} \in \mathbb{R}^{2d}$; $U(k) = (u(k), u'(k)) \in \mathscr{U} \times \mathscr{U}$.

If it is possible to find an initial state in Q and an appropriate choice of the strings $\{u_k\}, \{u'_k\}$ such that the orbit of (7) remains in Q, it means that the two strings of inputs give rise to the same output for the system (1). Conditions ensuring that the state is outside Q for some k will be sought to guarantee left invertibility. Similarly, if the state exits Q at least once every k iterations after a finite transient, then the system (1) is uniformly left invertible in time k.

Definition 15. *Define $P_{(i,j)} = p_i p_j$. Given a compact set $K \subset \mathbb{R}^d$, the IFS with probabilities associated to the system (7) is*

$$\{K \times K; F_{(1,1)}, F_{(1,2)}, \ldots, F_{(m,m)}; P_{(1,1)}, P_{(1,2)} \ldots, P_{(m,m)}\}. \tag{8}$$ ◇

Thanks to Theorem 1, given a system of type (1) and a subset of input symbols C for the corresponding system of type (7), it is possible to describe a set $\mathscr{A}_C$ that is both an attractor and an invariant set.

Note that the attractor associated to a single $U \in \mathscr{U} \times \mathscr{U}$, indicated by X_U, by Contraction Theorem [14], is a *unique* fixed point, and it can be approximated iterating the map F_U. For every $U \in \mathscr{U} \times \mathscr{U}$, for all $X \in \mathbb{R}^{2d}$, let $X_U = \lim_{k \to \infty} F_U^k(X)$. The relative position of these fixed points with respect to Q is sufficient to conclude about the μ-AE left invertibility. Let Δ denote the diagonal of $\mathscr{U} \times \mathscr{U}$, i.e. $\Delta = \{(1,1), (2,2), \ldots, (m,m)\}$.

Theorem 2. *If there exists $U \notin \Delta$ such that $X_U \subset Q$, then the system (1) is not LI. If every $X_U, U \notin \Delta$ is not in $\bar{Q}$, the closure of Q, the system (1) is μ-AE LI.* ◇

Proof: Suppose that there exists $U \notin \Delta$ such that $X_U \subset Q$. Select X_U as initial condition and choose σ to be the constant sequence $\sigma_i = U$ $\forall i \in \mathbb{N}$. The resulting orbit is the constant orbit $X(i) = X_U$ $\forall i \in \mathbb{N}$, and it is clearly contained in Q, so the system is not LI, since $U \notin \Delta$.

Suppose that $U \in \mathscr{U} \times \mathscr{U}, U \notin \Delta \Rightarrow X_U \notin \bar{Q}$. First observe that in this hypothesis no attractor $\mathscr{A}_C, C \not\subset \Delta$, is included in $\bar{Q}$, because of Proposition 1: indeed every attractor $\mathscr{A}_C, C \not\subset \Delta$, must contain a $X_U, U \notin \Delta$.

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Then Theorem 1 assures that for μ-almost every couple of input strings the trajectory is dense in $\mathscr{A}$. So, if $\mathscr{A}$ has a point p not in $\bar{Q}$, the generic trajectory contains points arbitrarily close to p and so contains points that are not included in Q. This proves the μ-AE left invertibility result. $\diamondsuit$

Remark 3. *As already observed in Remark 2, μ-almost everywhere left invertibility does not depend on the probabilities p_i's.* $\diamondsuit$

We now introduce a graph, whose properties are linked to left invertibility.

Definition 16. *The graph G_k of depth k associated to the attractor $\mathscr{A}$ is given by:*
• *The set of vertices $V = \{\mathscr{A}_{\sigma_1 \ldots \sigma_k} = F_{\sigma_k} \circ \ldots \circ F_{\sigma_1}(\mathscr{A}) : \sigma_i \in \Sigma\}$*
• *There is an edge from $\mathscr{A}_{\sigma_1 \ldots \sigma_k}$ to $\mathscr{A}_{\omega_1 \ldots \omega_k}$ if and only if $\sigma_{i+1} = \omega_i$, for $i = 1, \ldots, k-1$. In this case we say that the edge is induced by the input ω_k.* $\diamondsuit$

Remark 4. *It follows from the definition of G_k that*
    (1) If there is an edge between $\mathscr{A}_{\sigma_1 \ldots \sigma_k}$ and $\mathscr{A}_{\omega_1 \ldots \omega_k}$, then there exists $U \in \mathscr{U} \times \mathscr{U}$ such that $F_U(\mathscr{A}_{\sigma_1 \ldots \sigma_k}) \subset \mathscr{A}_{\omega_1 \ldots \omega_k}$.
    (2) $\bigcup_{\sigma_i \in \Sigma} \mathscr{A}_{\sigma_1 \ldots \sigma_k} = \mathscr{A}$; i.e. the union of vertexes of G_k, considered as sets, is the whole attractor.
    (3) If the attractor $\mathscr{A}$ is totally disconnected, the vertices of G_k provide a partition of the attractor. $\diamondsuit$

Definition 17. *Consider the graph G_k, and collapse to a single vertex, denoted by $\mathscr{A}_I$, all vertices $\mathscr{A}_{\sigma_1 \ldots \sigma_k}$ such that $\mathscr{A}_{\sigma_1 \ldots \sigma_k} \cap \{\mathbb{R}^{2d} \setminus Q\} \neq \emptyset$. Moreover every edge from $\mathscr{A}_I$ is deleted. This new graph is called internal invertibility graph, and denoted with IG_k. The set of vertices of IG_k is denoted by V_{IG_k}.*

Consider the graph G_k, and collapse to a single vertex, denoted by $\mathscr{A}_E$, all vertices $\mathscr{A}_{\sigma_1 \ldots \sigma_k}$ such that $\mathscr{A}_{\sigma_1 \ldots \sigma_k} \cap Q = \emptyset$. This new graph is called external invertibility graph, and denoted by EG_k. The set of vertices of EG_k is denoted by V_{EG_k}.

As done with G_k, in the following we use the symbols V_{IG_k}, V_{EG_k} to denote the vertices of the graphs as well as the set of points that they represent. $\diamondsuit$

Definition 18. *A path $\{V_i\}_{i=0}^{\infty}$ on EG_k or IG_k is called proper path if the first edge is induced by an input not in Δ.* $\diamondsuit$

Proposition 3. *There exists an orbit of the system (7) included in the set of vertices of $V_{IG_k} \setminus \mathscr{A}_I$, if and only if there exists an infinite path in IG_k.*

Proof: If there is an orbit $\{X_i\} = \{F_{\sigma_i} \circ \ldots \circ F_{\sigma_1}(X_0)\}_{i=1}^{\infty}$ included in $V_{IG_k} \setminus \mathscr{A}_I$, then we can construct an infinite path on IG_k by associating to each X_i, $i \geq k$ the vertex $\mathscr{A}_{\sigma_{i-k}, \ldots, \sigma_{i-1}}$: it is a path (of infinite length) on the graph IG_k, and never touches $\mathscr{A}_I$.

Conversely, if there exists an infinite path $\{V_1, V_2, \ldots\}$ in IG_k, first note that it cannot touch $\mathscr{A}_I$ because there is no edge starting from $\mathscr{A}_I$. Then, thanks to the first point of Remark 4, it is possible to exhibit an orbit included in $V_{IG_k} \setminus \mathscr{A}_I$. Indeed if any $X_1 \in V_1$ is chosen, then there exists an $\sigma_1 \in \mathscr{U} \times \mathscr{U}$ such that $X_2 = F_{\sigma_1}(X_1) \in V_2$; there exists $\sigma_2 \in \mathscr{U} \times \mathscr{U}$ such that $X_3 = F_{\sigma_2}(X_2) \in V_3$. Continuing in this way for every $i \in \mathbb{N}$ it is found an $\sigma_{i-1} \in \mathscr{U} \times \mathscr{U}$ such that $X_i = F_{\sigma_{i-1}}(X_{i-1}) \in V_i$. This procedure gives rise to an orbit of the system (7) included in $V_{IG_k} \setminus \mathscr{A}_I$. $\diamondsuit$

Proposition 4. *Fix $i \in \mathbb{N}$. A sufficient condition for the uniform left invertibility of the system (7) is the absence of an infinite proper path on EG_i.*

Proof: Suppose that the system (1) is not ULI. Then, for every $j \in \mathbb{N}$ there exists an orbit $\{X_p\} = \{F_{\sigma_p} \circ \ldots \circ F_{\sigma_1}(X_0)\}_{p=1}^{\infty}$ such that $\sigma_1 \notin \Delta$ and $\{X_p\}_{p=0}^{j} \subset Q \cap \mathscr{A}$. Then we can construct a path on EG_i by associating at each X_p, $p \geq i$ the vertex $\mathscr{A}_{\sigma_{p-i}, \ldots, \sigma_{p-1}}$ such that $X_p \in \mathscr{A}_{\sigma_{p-i}, \ldots, \sigma_{p-1}}$. It is a finite proper path on the graph EG_i thanks to the Remark 4,

and for every $p \leq j$ $V_p \neq \mathscr{A}_E$, because $\mathscr{A}_E \cap Q = \emptyset$ and $\{X_p\}_{p=0}^{j} \subset Q$. It remains to note that, since this construction can be made for every $j \in \mathbb{N}$ and, since the invertibility graph is finite, there is an infinite proper path in EG_i. $\diamond$

Proposition 5. *Suppose that no point of $\mathscr{A}$ belongs to the boundary ∂Q of Q, or equivalently that $\inf_{a \in \mathscr{A}} d(a, \partial Q) > 0$. Then there exists a $k \in \mathbb{N}$ such that $V_{IG_k} \setminus \mathscr{A}_I = V_{IG_k} \cap Q$. This in turn implies that $IG_k = EG_k$*

Proof: We first show that $\mathscr{A} \cap \partial Q = \emptyset$ if and only if $\inf_{a \in \mathscr{A}} d(a, \partial Q) > 0$.
Suppose that $\mathscr{A} \cap \partial Q = \emptyset$. If $\inf_{a \in \mathscr{A}} d(a, \partial Q) = 0$, then choose a sequence $\{a_k\} \subset \mathscr{A}$ such that $\lim_{k \to \infty} a_k \searrow 0$. Since $\mathscr{A}$ is compact there is an accumulation point $a \in \mathscr{A}$ for $\{a_k\}$. Then it is immediate to see that $d(a, \partial Q)$ must be zero. So a should belong to ∂Q because ∂Q is a closed set. This is impossible because we supposed that $\mathscr{A} \cap \partial Q = \emptyset$. So it must be $\inf_{a \in \mathscr{A}} d(a, \partial Q) > 0$. Conversely if $\inf_{a \in \mathscr{A}} d(a, \partial Q) > 0$, then $\mathscr{A} \cap \partial Q = \emptyset$.
So, assume now that $c = \inf_{a \in \mathscr{A}} d(a, \partial Q) > 0$. Choose k such that every set $V_{IG_k}^i$ has a diameter $\delta_i < c$. Then $V_{IG_k} \setminus \mathscr{A}_I = V_{IG_k} \cap Q$ because no V_{IG_k} can intersect ∂Q. $\diamond$

Theorem 3. *Suppose that $\mathscr{A} \cap \partial Q = \emptyset$. Then the following conditions are equivalent:*

(1) The system (1) is LI;
(2) IG_k does not contain an infinite proper path, where k is such that $V_{IG_k} \setminus \mathscr{A}_I = V_{IG_k} \cap Q$;
(3) The system (1) is ULI;

Proof:

"1. $\Leftrightarrow$ 2."
Suppose that IG_k contains an infinite proper path $\{V_i\}_{i \in \mathbb{N}}$. By Proposition 3 there exists an orbit $\{X_i\}_{i \in \mathbb{N}}$, where $X_i \in V_i$. Note that $X_i \notin \mathscr{A}_I$ because no edges start from $\mathscr{A}_I$. The orbit is included in Q because $X_i \in V_{IG_k} \setminus \mathscr{A}_I = V_{IG_k} \cap Q$. Moreover $X_1 = F_U(X_0)$ with $U \notin \Delta$, because the path is proper. This contradicts left invertibility.
Viceversa, suppose that the system (1) is not left invertible. Then there exists an orbit $\{X_i\}_{i \in \mathbb{N}} \subset Q$, such that $X_i = F_U(X_{i-1})$ and $U \notin \Delta$ for an infinite number of $i \in \mathbb{N}$. This orbit, induces, in the same way as in the proof of Proposition 3, an infinite proper path in IG_k.
"2. $\Leftrightarrow$ 3."
Suppose that IG_k contains an infinite proper path of arbitrary length $\{V_i\}_{i=0}^{\infty}$. By Proposition 3 there exists an orbit $\{X_i\}_{i \in \mathbb{N}}$, where $X_i \in V_i$. The orbit is included in Q because $X_i \in V_{IG_k} \setminus \mathscr{A}_I = V_{IG_k} \cap Q$. Moreover $X_1 = F_U(X_0)$, with $U \notin \Delta$. This contradicts uniform left invertibility. The inverse implication is true by Proposition 4. $\diamond$

Remark 5. *The condition $\mathscr{A} \cap \partial Q = \emptyset$ is not very relevant from a practical point of view, especially when the partition $\mathscr{P}$ can be designed. In this case indeed the boundary of Q can be positioned in the complement of the attractor $\mathscr{A}$. $\diamond$*

Remark 6. *In the hypothesis of Proposition 5 the construction of internal and external invertibility graph allows to solve the left invertibility problem of joint contractive systems with computational methods, and the effective procedure to do this will be a part of future work (see section 7). $\diamond$*

Remark 7. *As already observed in Remark 2, left invertibility does not depend on the probabilities p_i's. $\diamond$*

3.1. **IFS techniques for joint expansive systems.** Suppose that the system (1) is joint expansive. Then $f_u(\cdot)$ admits an inverse for every $u \in \mathscr{U}$. It is so defined the following

system on $\mathbb{R}^{2d}$:

$$Z(k+1) = G_{U(k)}(Z(k)) = \begin{bmatrix} f_{u(k)}^{-1}(z_1(k)) \\ f_{u'(k)}^{-1}(z_2(k)) \end{bmatrix} \tag{9}$$

where $Z(k) = \begin{bmatrix} z_1(k) \\ z_2(k) \end{bmatrix} \in \mathbb{R}^{2d}$; $U(k) = (u(k), u'(k)) \in \mathscr{U} \times \mathscr{U}$;

Since system (9) is joint contractive, for every $C \subset \mathscr{U} \times \mathscr{U}$ there exists an attractor $\mathscr{S}_C \subset \mathbb{R}^{2d}$. The set $\mathscr{S}_C$ is formed of all initial conditions giving rise to bounded orbits of system (7) with inputs in C:

Proposition 6. *Let $\mathscr{S}_C \subset \mathbb{R}^{2d}$ be the attractor obtained from the joint contractive system (9). If $\{X_k\}$ is an infinite bounded orbit of the system (7) with inputs in C, then $\forall k \in \mathbb{N}$ $X_k \in \mathscr{S}_C$.*

Proof: Call $\{U_k\}_{k \in \mathbb{N}} \subset C$ the input sequence of system (7) that gives rise to the orbit $\{X_k\}$. Then, for every $i \in \mathbb{N}$ there exists a point $s(i) \in S_C$ (i.e. any point in S_C with address $(U_0, \ldots, U_{i-1})$) such that the orbit $\{X_k'(i)\}_0^\infty$ of the system (7) with input sequence $\{U_k\}_0^\infty$ belongs to S_C for $k = 0, \ldots, i$. Let us consider, for every $j \leq i$, $d(X_j, X_j'(i))$.

Since the system (7) is joint expansive there exists a $p \in \mathbb{N}$ and a $c > 1$ such that for every $j \leq k \leq i$, and for every $p \leq k \leq i$

$$d(X_k, X_k'(i)) \geq c^{\lfloor (k-j)/p \rfloor} d(X_j, X_j'(i)).$$

The latter equation implies that, if there exists $\varepsilon > 0$ such that $d(X_j, X_j'(i)) > \varepsilon$ for infinitely many $i \in \mathbb{N}$, then $\{X_k\}_{k=0}^\infty$ would be unbounded. In fact $\|\{X_k'(i)\}_{k=0}^i\| \leq \max_{s \in \mathscr{S}_\mathscr{C}} \|s\|$ for every $k, i \in \mathbb{N}$ with $k \leq i$. Therefore

$$d(X_j, \mathscr{S}_C) = 0,$$

and, since $\mathscr{S}_C$ is compact we deduce that $X_j \in \mathscr{S}_C$. $\diamondsuit$

Theorem 4. *Suppose that the system (1) is joint expansive. If the system is LI then $\mathscr{S}_U \not\subset Q$ for every $U \in \mathscr{U} \times \mathscr{U}$, with $U \notin \Delta$.*
Moreover, if we restrict our attention to bounded orbits of system (1), and $\mathscr{S}_U \not\subset \overline{Q}$ for every $U \in \mathscr{U} \times \mathscr{U}$, with $U \notin \Delta$, then the system is $\mu-AE$ LI, and Theorems 2, 3 apply.

Proof: The necessary condition of the first part of the Theorem is obvious, and follows from the same reasoning as in the proof of Theorem 2. The second part follows from Proposition 6. $\diamondsuit$

4. A TEST FOR LEFT INVERTIBILITY IN JOINT CONTRACTIVE SYSTEMS

The application of Theorem 3, though computationally possible, can be very hard, especially in presence of complicated attractors or attractors near to the boundary of the set Q. So we suggest a low-complexity procedure to test left invertibility, based on the implementation of Random Iteration Algorithm [1, 10].

Let us briefly illustrate how the Random Iteration Algorithm proceeds. An initial point $x_0 \in \mathbb{X}$ is chosen. One of the transformations is selected "at random" from the set $\{F_1, \ldots, F_n\}$, but the probability that each F_i is selected is p_i for $i = 1, \ldots, n$. The selected transformation is applied to produce a new point $x_1 \in \mathbb{X}$. Again a transformation is selected using associated probabilities, in the same manner, independently from the previous choice, and applied to x_1 to produce a new point x_2, and so on. The implementation of Random Iteration Algorithm with any initial condition let the orbit tend to the attractor of the system [1, 10].

Definition 19. *Let $(\mathbb{X}, d)$ be a compact metric space, and let $\mathbb{M}(\mathbb{X})$ denote the space of normalized Borel measures on $\mathbb{X}$. The Hutchinson metric d_H on $\mathbb{M}(\mathbb{X})$ is defined by*

$$d_H(\nu, \lambda) = \sup \Big\{ \int_{\mathbb{X}} f d\nu - \int_{\mathbb{X}} f d\lambda :$$

$$\left. f : \mathbb{X} \to \mathbb{R} \ continous, |f(x) - f(y)| \le d(x,y) \right\}$$

for every $\nu, \lambda \in \mathbb{M}(\mathbb{X})$. $\Diamond$

Theorem 5. [17] *Let $(\mathbb{X}, d)$ be a compact metric space, and let $\mathbb{M}(\mathbb{X})$ denote the space of normalized Borel measures on $\mathbb{X}$. Then $(\mathbb{M}(\mathbb{X}), d_H)$ is a compact metric space. $\Diamond$*

Definition 20. *Let $(\mathbb{X}, d)$ be a compact metric space, and let $\mathbb{M}(\mathbb{X})$ denote the space of normalized Borel measures on $\mathbb{X}$. Let $\{\mathbb{X}, F_1, \ldots, F_n, p_1, \ldots, p_n\}$ be an IFS. The Markov operator associated with the IFS is the function $M : \mathbb{M}(\mathbb{X}) \to \mathbb{M}(\mathbb{X})$ defined by*

$$M(\nu) = \sum_{i=1}^{n} p_i \nu \circ F_i^{-1}$$

for every $\nu \in \mathbb{M}(\mathbb{X})$. $\Diamond$

Theorem 6. [1] *Let $(\mathbb{X}, d)$ be a compact metric space and let $\{\mathbb{X}, F_1, \ldots, F_n, p_1, \ldots, p_n\}$ be an IFS with probabilities with contractivity factor s. Let $M : \mathbb{M}(\mathbb{X}) \to \mathbb{M}(\mathbb{X})$ be the associated Markov operator. Then M is a contraction mapping, with contractivity factor s, with respect to the Hutchinson metric on $\mathbb{M}(\mathbb{X})$. That is, for every $\nu, \lambda \in \mathbb{M}(\mathbb{X})$*

$$d_H(M(\nu), M(\lambda)) \le s d_H(\nu, \lambda).$$

In particular, there is a unique measure υ such that $M\upsilon = \upsilon$. $\Diamond$

Definition 21. *The fixed point υ of the Markov operator M is called the invariant measure of the IFS. $\Diamond$*

Theorem 7. [12]
Let $\{\mathbb{X}, F_1, \ldots, F_n, p_1, \ldots, p_n\}$ be an IFS, and indicate with $\{x_k\}_{k=0}^{\infty}$ a generic orbit of the IFS. Let B be a Borel subset of $\mathbb{X}$ with boundary of zero Lebesgue measure. Let $\mathcal{N}(B,k)$ be the number of points in $\{x_0, \ldots, x_k\} \cap B$. Let υ the invariant measure of the IFS. Then, for all initial conditions x_0 and μ-almost every $\sigma \in \Sigma$

$$\upsilon(B) = \lim_{k \to \infty} \left\{ \frac{\mathcal{N}(B,k)}{k+1} \right\}. \Diamond \tag{10}$$

Proposition 7. *In the hypotheses and notations of Theorem 7, suppose that $i \in \mathbb{N}$ and let $\mathcal{N}(B, i, k)$ be the number of points in $\{x_i, x_{2i}, \ldots, x_{ki}\} \cap B$. Then, for all initial conditions x_0, for all $i \in \mathbb{N}$, and μ-almost every $\sigma \in \Sigma$*

$$\upsilon(B) = \lim_{k \to \infty} \left\{ \frac{\mathcal{N}(B, i, k)}{k+1} \right\}. \tag{11}$$

   Proof: We will show that, if we consider the IFS

$$\left\{ \mathbb{X}; \{F_{j_1} \circ \ldots \circ F_{j_i}\}; \{p_{j_1} \cdot \ldots \cdot p_{j_i}\} \right\} \tag{12}$$

where $j_1, \ldots, j_i \in \{1, \ldots, n\}$, then υ is the fixed point of the Markov operator. So by Theorem 7 the equation (11) holds. Indeed

$$\sum_{j_1, \ldots, j_i \in \{1, \ldots, n\}} p_{j_1} \cdot \ldots \cdot p_{j_i} \upsilon (F_{j_1} \circ \ldots \circ F_{j_i})^{-1} =$$

$$= \sum_{j_1, \ldots, j_i \in \{1, \ldots, n\}} p_{j_1} \cdot \ldots \cdot p_{j_i} \upsilon F_{j_i}^{-1} \circ \ldots \circ F_{j_1}^{-1} =$$

$$= \sum_{j_1} p_{j_1} \cdot \ldots \cdot \underbrace{\sum_{j_i} p_{j_i} \upsilon F_{j_i}^{-1}}_{= \upsilon} \circ \ldots \circ F_{j_1}^{-1} = \upsilon$$

because of the invariant property of υ. $\Diamond$

   Theorem 7 and Proposition 7 give more information about the dynamics inside the attractor. They also suggest the use of Random Iteration Algorithm to have an approximation

of the attractor of the system (7), and an estimate of the number of steps necessary to the uniform left invertibility (k in the definition of uniform left invertibility). The main steps of a possible strategy to use Random Iteration Algorithm to detect uniform left invertibility are the following:

• If any X_U, $U \notin \Delta$ is included in Q the system is not LI by Theorem 2. Otherwise the system is μ-AE LI (Theorem 2). Set the waiting time l to be the number of instants necessary to take the state near enough to the attractor. For any fixed compact set K, we can find such an $l \in \mathbb{N}$ depending only on K, because of the contraction hypothesis.

• After l instants we can start to "identify" the real orbit with a fictitious one inside the attractor. This is possible because the distance between the state of the real orbit and fictitious one tends to zero, since the maps are contractions.

• Estimate $\nu(\mathscr{A} \cap Q)$: applying the Random Iteration Algorithm the limit is obtained with probability one with respect to the measure μ, by Theorem 7 and Proposition 7. We expect the state to exit Q every $\lceil 1/\nu(\mathscr{A} \cap Q) \rceil$ in average and with probability one.

Remark 8. *Consider, for $k \in \mathbb{N}$, the vertex $\mathscr{A}_E(k)$ of the external invertibility graph. Then*

$$\mu[\mathscr{A}_E(k)] = \sum_{\mathscr{A}_{\sigma_1 \dots \sigma_k} \subset \mathscr{A}_E(k)} P_{\sigma_1} \cdot \dots \cdot P_{\sigma_k}.$$

If RIA runs for s steps, we obtain $s - k + 1$ strings of length k. So, computing (see Theorem 7)

$$\frac{\mathscr{N}(\mathscr{A}_E(k), s)}{s + 1},$$

we achieve an estimate of $\mu[\mathscr{A}_E(k)]$, that theoretically does not depend on the cardinality of the input set (we can choose for instance $s = 10000$).

This, together with the fact that $\mathscr{A}_E(k)$ approximates $\mathscr{A} \setminus Q$ with a geometrical rate with respect to k, allows us to obtain an estimate of the average of the time within which the state exits Q. $\diamondsuit$

5. AN ALGORITHM TO DETECT INPUTS IN JOINT CONTRACTIVE SYSTEMS

Let a joint contractive system of type (1) be given. We want to develop an algorithm that recover the input sequences from the output ones. This algorithm relies on two basic assumptions: the joint contractivity hypothesis on the system, and the possibility of determining *a priori* the uniform left invertibility.

Let the contraction factor be c, suppose that the hypotheses of Theorem 3 hold, and that the system is ULI in i steps. We assume that a bounded estimate of the state is possible. We consider then, without loss of generality, 0 to be the first instant for which the distance between the state in the system (7) and the attractor $\mathscr{A}$ is less than $\frac{d(\mathscr{A}, \partial Q)}{2}$. So we suppose that $x_0 \in I$, where I is a bounded set included in only one element of the partition $\mathscr{P}$. The following are the main steps of the algorithm.

(1) Choose $\hat{x}_0$ to be a point of I, and R to be the smallest radius of a ball $B(\hat{x}_0, R)$ with center in $\hat{x}_0$ such that $I \subset B(\hat{x}_0, R)$. Moreover construct an $\varepsilon-$grid of $I \cap B(\hat{x}_0, R)$, with $\varepsilon < \frac{d(\mathscr{A}, \partial Q)}{2}$.

(2) Compute the output symbols of the system of length i with $\hat{x}_0$ as initial condition and all possible input strings of length i:

$$S = \left\{ F_0^i\left(\hat{x}_0, u(1), \dots, u(i)\right) : u(k) \in \mathscr{U} \right\}.$$

Compare these output strings with the string $F_0^i\left(x_0, \bar{u}(1), \dots, \bar{u}(i)\right)$ produced by the system.

(3) • If $F_0^i\left(x_0, \bar{u}(1), \dots, \bar{u}(i)\right) = F_0^i\left(\hat{x}_0, u(1), \dots, u(i)\right) \in S$, then, since uniform left invertibility of the system holds, we are sure that $u(1) = \bar{u}(1)$, i.e. $u(1)$ is actually the input used by the system. Go to the step 4.

- If $F_0^i\left(x_0, \bar{u}(1), \ldots, \bar{u}(i)\right) \notin S$ then replace $\hat{x}_0$ with another point in the ε−grid of $I \cap B(\hat{x}_0, R)$, call it again $\hat{x}_0$ and go to the step 2.

(4) Set $B(f_{\bar{u}(1)}\hat{x}_0, cR) \cap \mathscr{P}_1 = I$, where $\mathscr{P}_1$ is element of the partition $\mathscr{P}$ to which $x_1 = f_{\bar{u}(1)}x_0$ belongs, and go to the step 1.

There are two important things to be observed. The first one concerns the step 3 of the algorithm. There always exists an $\hat{x}_0$ in the ε−grid of $I \cap B(\hat{x}_0, R)$ such that the first point of step 3 works: that's because the hypothesis of Theorem 3 holds and the system is ULI in i steps. The second remark is that for every time the algorithm apply the steps $1, 2, 3$ a number of time such that all composite maps are contractive, the estimate of the state is made more precise up to a factor $c < 1$. As a consequence, an asymptotic estimate (with geometric rate) of the state is possible.

6. EXAMPLES

Example 4. *Consider the I/O quantized linear system (2), with*

$$d = 5, \quad A = \begin{pmatrix} \frac{1}{10} & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{10} & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{10} & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{10} & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{10} \end{pmatrix} = J_5(\frac{1}{10}), \quad B = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \mathbf{1}_5$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \end{pmatrix} = \pi_1(5), \quad q(\cdot) = \lfloor \cdot \rfloor, \quad \mathscr{U} = \{0, 1/20, 1\}.$$

The system is joint contractive (see Example 3). We have, for example, $X_{(0,1/20)} \in Q$, so the system is not LI by Theorem 2.
Consider instead $\mathscr{U} = \{0, 1\}$. Then $X_{(0,1)} \notin Q$ and $X_{(1,0)} \notin Q$. By Theorem 2 the system is μ−AE LI. We are going to show that the system is indeed ULI.
Define

$$U_{(i,j)} = \begin{pmatrix} \mathbf{1}_5 u_i \\ \mathbf{1}_5 u_j \end{pmatrix}, \quad I_i = [0, i), \quad M = \begin{pmatrix} J_5(\frac{1}{10}) & 0 \\ 0 & J_5(\frac{1}{10}) \end{pmatrix},$$

$$S = \left(I_2 \times I_4 \times I_6 \times I_8 \times I_{10}\right) \times \left(I_2 \times I_4 \times I_6 \times I_8 \times I_{10}\right).$$

Direct calculations show that

$$\bigcup_{(i,j)\in\{0,1\}^2} M(S) + U_{(i,j)} \subset S; \tag{13}$$

$$U_{(i,j)} \neq U_{(i',j')} \Rightarrow M(S) + U_{(i,j)} \bigcap M(S) + U_{(i',j')} = \emptyset; \tag{14}$$

$$(i,j) \notin \Delta \Rightarrow M(S \cap Q) + U_{(i,j)} \bigcap Q = \emptyset. \tag{15}$$

Equation (13) implies that S contains the attractor, equation (14) implies, by Proposition 2, that the attractor is totally disconnected, and equation (15) implies by Theorem 3 that the system is uniformly left invertible in 1 step. ◇

Let's give now a nonlinear example with an expansive system.

Example 5. *Take $\mathbb{X} = \mathbb{R}$, $\mathscr{U} = \{1, 2\}$ and*

$$f_1(x) = 2x - \frac{1}{2} \qquad f_2(x) = \begin{cases} 3x + x^2 & for\ x \geq 0 \\ 3x - x^2 & for\ x < 0 \end{cases} \tag{16}$$

The two maps are expansive: f_1 is an affine linear map, and clearly $d(f_1(x_1), f_1(x_2)) = 2d(x_1, x_2)$. To show that f_2 is expansive compute its derivative: $f'(x) = 3 + 2|x| \geq 3$. It's easy to see that if a function has derivative greater than $1 + \varepsilon$, for some $\varepsilon > 0$, on all $\mathbb{R}$, then it is expansive. Moreover f_1 and f_2 are bijective and their inverse are

$$f_1^{-1}(z) = \frac{z + \frac{1}{2}}{2} \qquad f_2^{-1}(z) = \begin{cases} \frac{-3+\sqrt{9+4z}}{2} & for\ z \geq 0 \\ \frac{3-\sqrt{9-4z}}{2} & for\ z < 0 \end{cases}$$
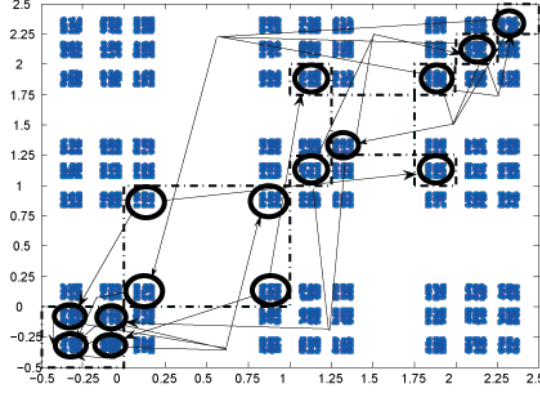
FIGURE 1. Attractor of system (17), with all internal edges of IG_2: Q is drawn with dashed-dotted lines.

If a bounded orbit is observed in the system of type (7) relative to (16), then this orbit is included in $[0, \frac{1}{2}] \times [0, \frac{1}{2}]$, since this set includes the attractor of the system of type (9) relative to (16), because

$$f_1^{-1}\left([0, \frac{1}{2}]\right) \subset [0, \frac{1}{2}], \quad f_2^{-1}\left([0, \frac{1}{2}]\right) \subset [0, \frac{1}{2}].$$

So by Theorem 4 that the system given by (16) is not LI. ◇

Finally, we give another example in dimension 1, drawing the attractor of the system and the invertibility graph.

Example 6. *Consider an I/O quantized linear system with*

$$d = 1, \ a = 1/5, \ b = c = 1, \ \mathscr{U} = \{-0.3, 0.9, 1.9\}. \tag{17}$$

We consider the partition q given by:

$$\begin{aligned}
&q(x) = \lfloor x \rfloor \ \ if \ x < 1; \\
&q(x) = 1 \ \ if \ x \in [1, 1.25[\cup[1.75, 2[\\
&q(x) = 2 \ \ if \ x \in [1.25, 1.75[\\
&q(x) = 3 \ \ if \ x \in [2, 2.25[\\
&q(x) = 4 \ \ if \ x \in [2.25, 3[\\
&q(x) = \lfloor x \rfloor + 2 \ \ if \ x \geq 3
\end{aligned} \tag{18}$$

System (17) is contractive because $|a| < 1$. The attractor of the system, in Figure 1, has been drawn with the Random Iteration Algorithm. Calculations show that it is included in the square $\mathscr{S} = [-1/2, 5/2] \times [-1/2, 5/2]$, and that

$$\left\{1/5 \cdot \mathscr{S} + (u_1, u_2)\right\} \bigcap \left\{1/5 \cdot \mathscr{S} + (u_3, u_4)\right\} = \emptyset$$

if $(u_1, u_2) \neq (u_3, u_4) \in \mathscr{U} \times \mathscr{U}$. This suffices, by Proposition 2, to conclude that the attractor of system (17) is totally disconnected. To our purpose we can divide the attractor in 81 parts (of which those included in Q are indicated by a circle around them), each one being represented by an address of two symbols in the alphabet $\mathscr{U} \times \mathscr{U}$. Then direct calculations shows that IG_2 and EG_2 coincide, and that the resulting graph is that one of figure 2, where edges induced by an input not int Δ are drawn with dashed arrows.

Clearly there does not exist proper paths of length greater than 2 in IG_2, so system (17) is uniformly left invertible in 2 steps. ◇
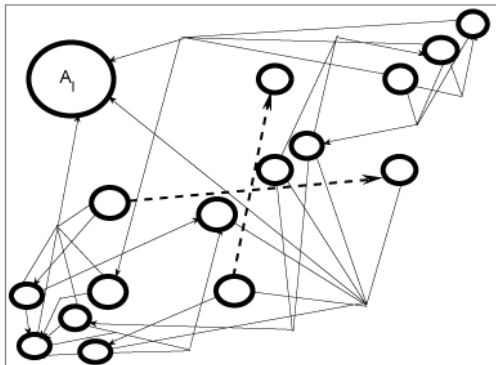
FIGURE 2. IG_2 of system (17)

7. CONCLUSIONS

In this paper we introduced a technique relating the theory of IFS to the left invertibility problem, for joint contractive dynamics. A necessary and sufficient condition for invertibility with probability one with respect to input strings is given (Theorem 2), and necessary and sufficient conditions for invertibility and uniform invertibility are stated (Theorem 3). In particular we showed that invertibility of joint contractive systems depends only on the properties of a compact set (the attractor). The attractor, from which the invertibility graph is obtained, is algorithmically approximable within a given, arbitrary small threshold, for example by the so called deterministic algorithm described in [1]. The implementation of an algorithm to construct the invertibility graph and check left invertibility is therefore straightforward.

Future research will include the extension of the results to non-contractive dynamics. For the case of non-contractive linear systems, for instance, in an on-going work a weaker form of left invertibility is defined which can be checked by attractor-based techniques in a similar way as described in this paper.

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