Shortest Paths With Side Sensors

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***Abstract*—We present a complete characterization of shortest paths to a goal position for a vehicle with unicycle kinematics and a limited range sensor, constantly keeping a given landmark in sight. Previous work on this subject studied the optimal paths in case of a frontal, symmetrically limited Field–Of–View (FOV). In this paper we provide a generalization to the case of arbitrary FOVs, including the case that the direction of motion is not an axis of symmetry for the FOV, and even that it is not contained in the FOV. The provided solution is of particular relevance to applications using side-scanning, such as e.g. in underwater sonar-based surveying and navigation.**

I. Introduction

In several mobile robot applications, a vehicle with nonholonomic kinematics of the unicycle type, equipped with a limited range sensor systems, has to reach a target while keeping an environment landmark in sight. For example, in the Visual–Based control field the vehicle usually has an onboard monocular camera with limited Field–Of–View (FOV) and, subject to nonholonomic constraints on its motion, must move maintaining in sight one or more specified features of the environment. On the other hand, in the field of underwater surveying and navigation, a common task for Autonomous Underwater Vehicles (AUV) equipped with side sonar scanners is to detect and recognize objects (mines, wrecks or archeological find, etc.) on the sea bed (see e.g. [1], [2]). Side-scan sonar is a category of sonar systems that is used to efficiently create an image of large areas of the sea. Therefore, in order to recognize objects AUVs must move keeping them inside the limited range of the sensor.

Motivated by those application, in this paper we propose the study of optimal paths (shortest ones) for a nonholonomic vehicle moving in a plane to reach a target position while making so that a given landmark fixed in the plane is kept inside a planar cone moving with the robot.

The literature of optimal (shortest) paths stems mainly from the seminal work on unicycle vehicles with a bounded turning radius by Dubins [3]. Dubins has characterized the finite family of optimal paths for the particular vehicle while a complete optimal control synthesis for this problem has been reported in [4]. Later on, a similar problem with the car moving both forward and backward has been solved with different approaches in [5], [6]. In particular, in [7] the optimal control synthesis for the Reeds&Shepp car has been provided. Minimum wheel rotation paths in for differential-drive robots have been considered in [8]. More recently, also the problem of determining minimum time trajectory has been taken into account in [9], [10] and [11] for particular classes of robots, e.g. latter is on underwater robots. Finally, previous works on the same subject of this paper ([12], [13], [14]) have studied the optimal paths in case of a vehicle with a limited onboard camera but only with a symmetric FOV with respect to the forward direction of the robot. In this paper, we present a more general synthesis of shortest paths in case of side sensor systems, like side sonar scanners on UAVs, where the forward direction is not necessarily included inside the sensor range modeled as a cone centered on the vehicle. The impracticability of paths that point straight to the feature lead to a more complex analysis of the reduction to a finite and sufficient family of optimal paths by excluding particular types of path.

In the rest of the paper, we provide a complete optimal synthesis for the problem, i.e. a finite language of optimal control words, and a global partition of the motion plane induced by shortest paths, such that a word in the optimal language is univocally associated to a region and completely describes the constrained shortest path from any starting point in that region to the goal point.

II. Problem Definition

Consider a vehicle moving on a plane where a right-handed reference frame $\langle W \rangle$ is defined with origin in O_W and axes X_w, Z_w. The configuration of the vehicle is described by $\xi(t) = (x(t), z(t), \theta(t))$, where $(x(t), z(t))$ is the position in $\langle W \rangle$ of a reference point in the vehicle, and $\theta(t)$ is the vehicle heading with respect to the X_w axis (see fig. 1). We assume that the dynamics of the vehicle are negligible, and that the forward and angular velocities, $\nu(t)$ and $\omega(t)$ respectively, are the control inputs to the kinematic model. Choosing polar coordinates for the vehicle $\eta = [\rho\,\psi\,\beta]^T$ (see fig. 1), the kinematic model of the unicycle-like robot is

$$\begin{bmatrix} \dot\rho \\ \dot\psi \\ \dot\beta \end{bmatrix} = \begin{bmatrix} -\cos\beta & 0 \\ \frac{\sin\beta}{\rho} & 0 \\ \frac{\sin\beta}{\rho} & -1 \end{bmatrix} \begin{bmatrix} \nu \\ \omega \end{bmatrix}. \tag{1}$$

We consider vehicles with bounded velocities which can turn on the spot. In other words, we assume

$$(\nu, \omega) \in U, \tag{2}$$

with U a compact and convex subset of $\mathbb{R}^2$, containing the origin in its interior.

The vehicle is equipped with a rigidly fixed sensor system with a reference frame $\langle C \rangle = \{O_c, X_c, Y_c, Z_c\}$ such that the center O_c corresponds to the robot's center $[x(t), z(t)]^T$ and the forward sensor axis Z_c forms an angle Γ w.r.t the robot's

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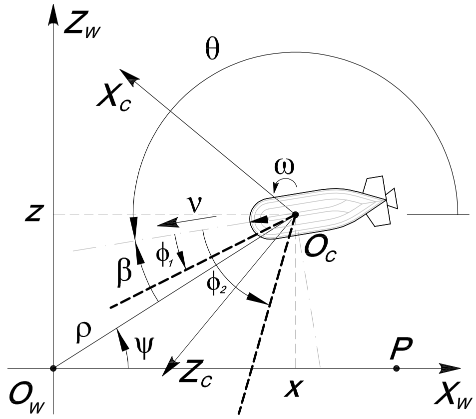


Fig. 1. Autonomous vehicle and systems coordinates. The vehicle's task is to reach P while keeping O_W within a limited SR (dashed lines).

forward direction. Moreover, let δ be the characteristic angle of the cone characterizing the limited SR and let us consider the most interesting problem in which $\delta \leq \pi/2$. Without loss of generality, we will consider $0 \leq \Gamma \leq \frac{\pi}{2}$, so that, when $\Gamma = 0$ the Z_c axis is aligned with the robot's forward direction (i.e. the particular case solved in [12]), whereas, when $\Gamma = \frac{\pi}{2}$, is aligned with the axle direction. Consider $\phi_1 = \Gamma - \frac{\delta}{2}$ and $\phi_2 = \Gamma + \frac{\delta}{2}$ the angles between the robot's forward direction and the right or left sensor's border w.r.t. Z_c axis, respectively. The restriction on $0 \leq \Gamma = \frac{\phi_1 + \phi_2}{2} \leq \frac{\pi}{2}$ will be removed at the end of this paper, and an easy procedure to obtain the subdivision for any value of Γ will be given.

Without loss of generality, we consider the position of the robot target point P to lay on the X_W axis, with coordinates $(\rho, \psi) = (\rho_P, \pi - \Gamma)$. We also assume that the feature to be kept within the SR is placed on the axis through the origin O_W and perpendicular to the plane of motion. We consider a planar SR with characteristic angle $\delta = |\phi_2 - \phi_1|$, which generates the constraints

$$\beta - \phi_1 \geq 0, \tag{3}$$

$$\beta - \phi_2 \leq 0. \tag{4}$$

Noticed that we place no restrictions on the vertical dimension of the SR. Therefore, the height of the feature on the motion plane, which corresponds to its Y_c coordinate in the sensor frame $\langle C \rangle$, is irrelevant to our problem. Hence, for our purposes, it is necessary to know only the projection of the feature on the motion plane, i.e. O_W.

The goal of this paper is to determine, for any point $Q \in \mathbb{R}^2$ in the robot space, the shortest path from Q to P such that the feature is maintained in the SR. In other words, we want to minimize the length of the path covered by the center of the vehicle under the *feasibility constraints* (1), (2), (3), and (4).

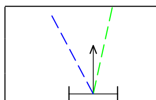
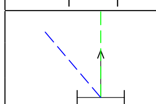
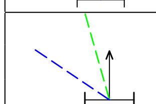
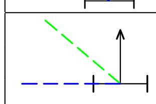
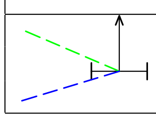
From the theory of optimal control with state and control constraints (see [15]) it is possible to show that, when constraints (3) and (4) are not active, extremals curves are straight lines (denoted by symbol S) and rotation on the spot (denoted by symbol $*$). On the other hand, when constraints (3) and (4) are active, the corresponding extremal maneuvers are two logarithmic spirals with characteristic angles ϕ_1 and ϕ_2 denoted by T_1 and T_2, respectively (see [12] for details).

Logarithmic spiral T with characteristic angle $\phi > 0$ ($\phi < 0$) rotates counterclockwise (clockwise) around the feature. We refer to counterclockwise and clockwise spirals as *Left* and *Right*, and by symbols T^L and T^R, respectively. The adjectives "left" and "right" indicate the half–plane where the spiral starts for an on–board observer aiming at the feature.

Notice that, for $\phi_2 = \pi/2$ the left SR border is aligned with the axle direction and the spiral T_2 becomes a circumference centered in O_W (denoted by C), whereas for $\phi_1 = 0$ the right SR border is aligned with the direction of motion and T_1 becomes an half line through O_W (denoted by H).

Extremal arcs can be executed by the vehicle in either forward or backward direction: we will hence use superscripts $+$ and $-$ to make this explicit (e.g., S^- stands for a straight line executed backward).

We will build extremal paths consisting of sequences of symbols, or *words*, in the alphabet $\mathscr{A} = \{*, S^+, S^-, E_1^+, E_1^-, E_2^+, E_2^-\}$, where the actual meaning of symbols depends on angles Γ and δ as in the following table.

	Frontal $0 \leq \Gamma < \frac{\delta}{2}$	$E_1 = T_1^L, E_2 = T_2^R$
	Borderline Frontal $\Gamma = \frac{\delta}{2}$	$E_1 = H, E_2 = T_2^R$
	Side $\frac{\delta}{2} < \Gamma < \frac{\pi-\delta}{2}$	$E_1 = T_1^R, E_2 = T_2^R$
	Borderline Side $\Gamma = \frac{\pi-\delta}{2}$	$E_1 = T_1^R, E_2 = C$
	Lateral $\frac{\pi-\delta}{2} < \Gamma \leq \frac{\pi}{2}$	$E_1 = T_1^R, E_2 = T_2^L$

Rotations on the spot ($*$) have zero length, but may be used to properly connect other maneuvers.

Let $\mathscr{L}_\Gamma$ be the set of possible words generated by the aforementioned symbols in $\mathscr{A}$ for each value of Γ. The rest of the paper is dedicated to showing that, due to the physical and geometrical constraints of the considered problem, a sufficient optimal finite language $\mathscr{L}_O \subset \mathscr{L}_\Gamma$ can be built such that, for any initial condition, it contains a word describing a path to the goal which is no longer than any other feasible path. Correspondingly, a partition of the plane in a finite number of regions is described, for which the shortest path is one of the words in $\mathscr{L}_O$.

III. Shortest path synthesis

In this section, we introduce the basic tools that will allow us to study the optimal synthesis of the whole state space of the robot, beginning from points on a particular sub–set of $\mathbb{R}^2$

such that the optimal paths are in a sufficient optimal finite language.

We start by noticing that the action of the alphabet $\mathscr{A}$ is invariant w.r.t. rotation and scaling. However, it is not invariant w.r.t. axial symmetry, as it happened in the related problem considered in [12].

Hence, we consider the following map f_Q:

Definition 1. *Given the target point $P = (\rho_P, 0)$ in polar coordinates, and $Q \in \mathbb{R}^2 \setminus O_W$, $Q = (\rho_Q, \psi_Q)$ with $\rho_Q \neq 0$, let $f_Q : \mathbb{R}^2 \to \mathbb{R}^2$ denotes the map*

$$f_Q(\rho_G, \psi_G) = \begin{cases} \left(\dfrac{\rho_G \rho_P}{\rho_Q}, \psi_G - \psi_Q\right) & \text{for } \rho_G \neq 0 \\ (0,0) & \text{otherwise.} \end{cases} \tag{5}$$

The map f_Q is the combination of a clockwise rotation by angle $\psi_G - \psi_Q$, and a scaling by a factor sρ_P/ρ_Q that maps Q in P.

Let γ be a path parameterized by $t \in [0,1]$ in the plane of motion $\gamma(t) = (\rho(t), \psi(t))$. Denote with $\mathscr{P}_Q$ the set of all feasible extremal paths from $\gamma(0) = Q$ to $\gamma(1) = P$.

Definition 2. *Given the target point $P = (\rho_P, 0)$ and $Q = (\rho_Q, \psi_Q)$ with $\rho_Q \neq 0$, let the* path transform *function F_Q be defined as*

$$\begin{aligned} F_Q : \mathscr{P}_Q &\to \mathscr{P}_{f_Q(P)} \\ \gamma(t) &\mapsto f_Q(\gamma(1-t)), \ \forall t \in I. \end{aligned} \tag{6}$$

Notice that $\tilde{\gamma}(t) = F_Q(\gamma(1-t))$ corresponds to $\gamma(t)$ transformed by f_Q and followed in opposite direction. Indeed, $\tilde{\gamma}$ is a path from $\tilde{\gamma}(0) = f_Q(P) = \left(\frac{\rho_P^2}{\rho_Q}, -\psi_Q\right)$ to $\tilde{\gamma}(1) = f_Q(Q) \equiv P$.

The F map has some properties that make it very useful to the study of our problem in a way which is to some extent similar to what described (for a different F map) in [12]. In particular, the locus of points Q such that $f_Q(P) = Q$, is the circumference with center in O_W and radius ρ_P. We will denote this circumference by $C(P)$ and the closed disk within $C(P)$ by $D(P)$.

$C(P)$ has an important role in the proposed approach since properties of F_Q will allow us to solve the synthesis problem from points on $C(P)$, and hence to extend the synthesis to $D(P)$ and to the whole motion plane. Indeed, $\forall Q \in C(P)$ and $\forall \gamma \in \mathscr{P}_Q$, $F_Q(\gamma) \in \mathscr{P}_{f_Q(P)}$ with $f_Q(P) \in C(P)$, i.e. a path from a point on $C(P)$ to P is mapped in a path from $C(P)$ to P.

Furthermore, F_Q transforms an extremal in $\mathscr{A}$ in itself but followed in opposite direction. Hence, F_Q maps extremal paths in $\mathscr{L}_\Gamma$ in extremal paths in $\mathscr{L}_\Gamma$. For example, let $w = S^- * H^- * S^+ * T_2^{R+}$ be the word that characterize a path from Q to P, the transformed path is of type $z = T_2^{R-} * S^- * H^+ * S^+$. With a slight abuse of notation, we will write $z = F_Q(w)$.

Proposition 1. *Given $Q \in \mathbb{R}^2$ and a path $\gamma \in \mathscr{P}_Q$ of length l, the length of the transformed path $\tilde{\gamma} = F_Q(\gamma)$ is $\tilde{l} = \frac{\rho_P}{\rho_Q} l$.*

The proof is easily obtained from a similar result in [12].

Based on the properties of F_Q, optimal paths from points on $C(P)$ completely evolve inside $C(P)$. To prove this statement we first show the following

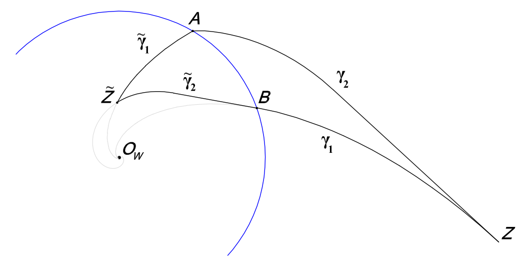


Fig. 2. Construction used in the proof of Theorem 1.

Theorem 1. *Given two points $A = (\rho_A, \psi_A)$ and $B = (\rho_B, \psi_B)$, with $\psi_A > \psi_B$ and $\rho = \rho_A = \rho_B$, and an extremal path γ from A to B such that for each point G of γ, $\rho_G > \rho$, there exists an extremal path $\tilde{\gamma}$ from A to B such that for each point $\tilde{G}$ of $\tilde{\gamma}$, $\rho_{\tilde{G}} < \rho$ and $\ell(\tilde{\gamma}) < \ell(\gamma)$.*

Proof: Consider a point $Z = (\rho_Z, \psi_Z)$ such that $\rho_Z = \max_{G \in \gamma} \rho_G > \rho$. Let γ_1 and γ_2 the sub–paths of γ from Z to B and from Z to A.

The sub–path γ_1, is rotated and scaled (contracted of factor $\frac{\rho}{\rho_Z} < 1$) such that Z is transformed in A obtaining a path $\tilde{\gamma}_1$ from A to $\tilde{Z} = (\frac{\rho^2}{\rho_Z}, \psi_A + \psi_B - \psi_Z)$. Similarly, γ_2, can be rotated and scaled with the same scale factor but different rotation angle w.r.t. γ_1 such that Z is transformed in B, see fig. 2. After geometrical considerations, it is easy to notice that the obtained path $\tilde{\gamma}_2$ starts in B and ends in $\tilde{Z}$.

The obtained paths are a contraction of γ_1 and γ_2 respectively and hence shorter. Moreover, any point G of γ_1 or γ_2 has $\rho_G > \rho$ hence is scaled in $\tilde{G}$ of $\tilde{\gamma}_1$ or $\tilde{\gamma}_2$ with $\rho_{\tilde{G}} = \frac{\rho \rho_G}{\rho_Z} < \rho$.

Concluding, we have obtained a shorter path from A to B that evolves completely in the disk of radius ρ. ■

An important but straightforward consequence of the theorem is the following:

Corollary 1. *For any path in $\mathscr{P}_Q$ with $Q \in C(P)$ there exists a shorter or equal-length path in $\mathscr{P}_Q$ that completely evolves in $D(P)$.*

IV. Optimal paths for points on $C(P)$

Our study of the optimal synthesis begins in this section addressing optimal paths from points on $C(P)$. We first need to establish an existence result of optimal paths.

Proposition 2. *For any $Q \in C(P)$ there exists a feasible shortest path to P.*

Proof: Because of state constraints (3), and (4), and the restriction of optimal paths in $D(P)$ (Corollary 1) the state set is compact. Furthermore, it is possible to give an upper-bound on the optimal path length for all $\Gamma \in [0, \frac{\pi}{2}]$. Indeed, given a point Q at distance ρ from O_W the optimal path to P is shorter or equal to the following paths based on the value of Γ and δ:

- Frontal ($0 \leq \Gamma \leq \frac{\delta}{2}$): $S^+ * S^-$ or $H^+ * H^-$ of length $\rho + \rho_P$;
- Side ($\frac{\delta}{2} < \Gamma < \frac{\pi - \delta}{2}$): $T_{1Q}^{R+} * T_{2P}^{R-}$, of length $\left(\frac{\rho - \rho_N}{\cos\phi_1} + \frac{\rho_P - \rho_N}{\cos\phi_2}\right)$, where N is the intersection point

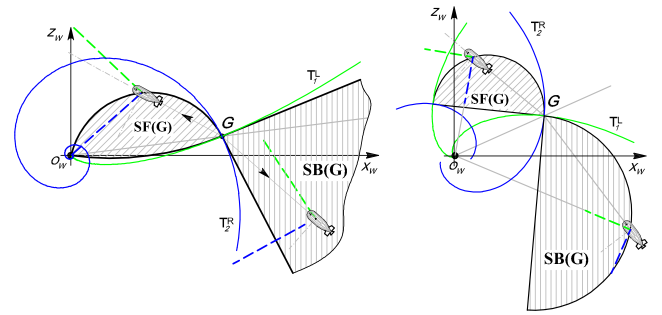


Fig. 3. Forward and backward straight path Regions from G for $0 \le \Gamma \le \frac{\delta}{2}$ on the left and $\frac{\pi-\delta}{2} \le \Gamma \le \frac{\pi}{2}$ on the right.

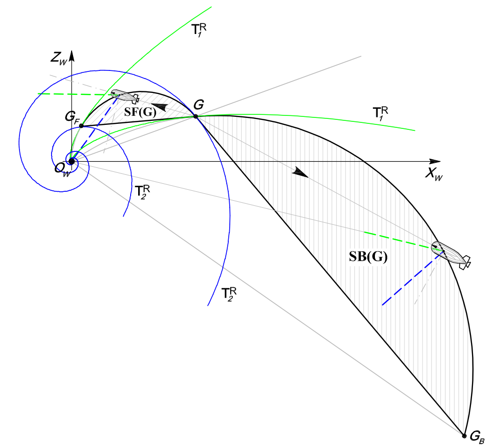


Fig. 4. Forward and backward straight path Regions from G for $\frac{\delta}{2} < \Gamma \le \frac{\pi-\delta}{2}$.

between spirals T^R_{1Q} and T^R_{2P} through Q and P respectively;

- Borderline Side ($\Gamma = \frac{\pi-\delta}{2}$: $T^{R+}_1 * C^-_P$) of length $\left(\frac{\rho-\rho_P}{\cos\phi_1} + (\psi_N - \psi_P)\rho_P\right)$, where N is the intersection point between spirals T^R_1 and C_P;
- Lateral ($\frac{\pi-\delta}{2} < \Gamma \le \frac{\pi}{2}$): $T^{L-}_{2Q} * T^{R-}_{1P}$, of length $\left(\frac{\rho-\rho_N}{\cos\phi_2} + \frac{\rho_P-\rho_N}{\cos\phi_1}\right)$, where N is the intersection point between spirals T^L_{2Q} and T^R_{1P}.

The system is also controllable because there always exists an intersection point between two spirals (even if degenerated in half–lines or circumferences) with different characteristic angle even if both clockwise or counterclockwise around the feature. Hence, Filippov existence theorem for Lagrange problems can be invoked [16]. ∎

In the following we provide a set of propositions that completely describe a sufficient optimal finite language for all values of Γ. However, for space limitations, proofs for some of the propositions are reported for the Side case only, i.e. $\frac{\delta}{2} < \Gamma < \frac{\pi-\delta}{2}$. Proofs for other cases can be easily obtained with analogous procedures.

Definition 3. *For any starting point $G = (\rho_G, \psi_G)$, let $SF(G)$ (resp. $SB(G)$) be the set of all points reachable from G with a forward (resp. backward) straight line without violating the SR constraints.*

We denote with $\partial SF_1(G)$ and $\partial SF_2(G)$ (resp. $\partial SB_1(G)$ and $\partial SB_2(G)$) the borders of $SF(G)$ (resp. $SB(G)$). Also, let $C_i(G)$ denote the circular arcs from G to O_W such that, $\forall V \in C_i(G)$, $\widehat{GVO_W} = \pi - |\phi_i|$ in the half-plane on the left of $\overline{GO_W}$ if $i = 1$, on the right if $i = 2$.

Remark 1. *For any starting point $G = (\rho_G, \psi_G)$, and for $\frac{\delta}{2} < \Gamma \le \frac{\pi-\delta}{2}$ (Side case), let S_{G_F} be the chord between G and $G_F = (\rho_G \frac{\sin\phi_1}{\sin\phi_2}, \psi_G + (\phi_2 - \phi_1)) \in C_2(G)$, i.e. such that $\widehat{O_W G G_F} = \phi_1$ (cf. fig. 4). Naming with C_{G_F} the arc between G and G_F, $SF(G)$ is the region between arc $\partial SF_2(G) = C_{G_F}$ and chord $\partial SF_1(G) = S_{G_F}$. Consider the rotation and scale that maps G_F in G and G in G_B: we have $\partial SB_1(G) = \partial SF_1(G_B)$, i.e. $\partial SB_2(G) = \partial SF_2(G_B)$. Notice that, also in this case, $SF(G)$ lays completely in the circle with center in O_W and radius ρ_G.*

As a consequence of Remark 1, $SF(G)$ is tangent in G to T^R_1 and T^R_2 (or C when $\Gamma = \frac{\pi-\delta}{2}$, i.e. $\phi_2 = \pi/2$). Moreover, $SF(G)$ is tangent in G_F to T^R_1 and T^R_2, see fig. 4.

Remark 2. *Forward and backward straight path Regions from G, for the other values of Γ, are represented in fig. 3 for space limitations.*

Remark 3. *Optimal forward (resp. backward) straight arcs from any G ends on $\partial SF(G)$ (resp. $\partial SB(G)$) (see [12] for details).*

Based on all the above properties, we are now able to obtain a sufficient family of optimal paths by excluding particular sequences of extremals.

Theorem 2. *Any path consisting in a sequence of a backward extremal arc followed by a forward extremal arc is not optimal.*

Proof: Observe that the distance from O_W is strictly increasing along backward extremal arcs (i.e. S^-, E^-_1, E^-_2 with $E_2 \neq C$) and strictly decreasing along forward extremal arcs (i.e. S^+, E^+_1, E^+_2 with $E_2 \neq C$). For continuity of paths, for any sequence of a backward extremal followed by a forward one, there exist points A and B that verify hypothesis of Theorem 1, hence it is not optimal.

Any sequence consisting in an extremal S (or E_1) of length ℓ and an extremal $E_2 = C$ (in any order and direction) is inscribed in two circumferences centered in O_W. Hence, the shortest sequence is the one with $E_2 = C$ along the circumference of smaller radius necessarily preceded by a forward S (or E_1) of same length ℓ.

Concluding, in an optimal path a forward arc cannot follow a backward arc. ∎

Theorem 3. *Any path consisting in a sequence of an extremal arc E_i and an extremal arc E_j followed in the same direction is not optimal for any $i, j = 1,2$ with $i \neq j$.*

Proof: for the Side case ($\frac{\delta}{2} < \Gamma < \frac{\pi-\delta}{2}$). By proving the non–optimality of $E^+_i * E^+_j$ the non–optimality of $E^-_j * E^-_i$ follows straightforward. Without loss of generality, we suppose $i = 1$ and $j = 2$. Let A and B be the initial and final

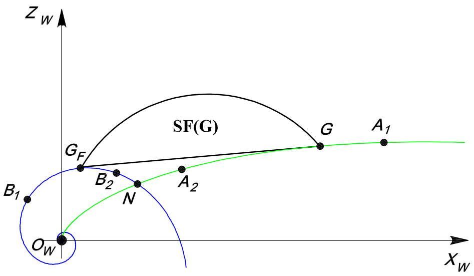


Fig. 5. Construction of a path shorter than $E_1^+ * E_2^+$ for $\frac{\delta}{2} < \Gamma \leq \frac{\pi-\delta}{2}$.

points of the path γ of type $E_1^+ * E_2^+$ and N the intersection points between E_1^+ and E_2^+. We now show for any value of Γ and δ that exists an sub–path of γ that can be shortened with a segment arc.

Referring to fig. 5, there exist $G \in E_1$ such that $SF(G)$ is tangent to E_1 in G and to E_2 in G_F. Positions of $A \in E_1$ and $B \in E_2$ with respect to G and G_F respectively generate three cases:

- $A = A_1$ and $B = B_1$, i.e. $\rho_B \leq \rho_{G_F} \leq \rho_N \leq \rho_G \leq \rho_A$. Obviously, γ can be shortened by $\overline{GG_F}$;
- $A = A_2$ and $B = B_1$, i.e. $\rho_B \leq \rho_{G_F} \leq \rho_N \leq \rho_A \leq \rho_G$. Consider $SF(A)$ and the point V of intersection between $\partial SF_2(A)$ and E_2, γ can be shortened by $\overline{AV}$. A similar procedure is used for $A = A_1$ and $B = B_2$, i.e. $\rho_{G_F} \leq \rho_B \leq \rho_N \leq \rho_G \leq \rho_A$;
- $A = A_2$ and $B = B_2$, i.e. $\rho_{G_F} \leq \rho_B \leq \rho_N \leq \rho_A \leq \rho_G$. In this case, $SF(A)$ and $SB(B)$ intersect E_2 and E_1 respectively.

■

Notice that the feasible sequences consisting of two extremals that we still need to discuss are those starting or ending with S followed in any direction. Indeed, it is obvious that E^+E^- and E^-E^+ are not optimal.

Proposition 3. *For any two points G, H, consider a spiral arc E_i $(i = 1, 2)$ from G to H, and denote by r_G, r_H the tangent lines to E_i in G and H, respectively. Let $A = r_G \cap r_H$. Then, the length of E_i is less than the sum of lengths of the segments $\overline{GA}$ and $\overline{AH}$.*

The proof of this proposition does not depend on Γ and can be found in [12].

Proposition 4. *Consider any two points G and H on a spiral arc E_i $(i = 1, 2)$. Let $\tilde{E}$ be the set of points between E_i and its symmetric w.r.t. $\overline{GH}$. A shortest path between G and H that evolves completely outside region $\tilde{E}$ is the arc of E_i between G and H.*

Proof: Let R be the quadrilateral circumscribed to $\tilde{E}$ tangent to E_i in G and H. Furthermore, let r_G and r_H the straight lines tangent to E_i in G and H, let N their point of intersection. Any path γ connecting G to H that evolves outside R is longer than the polygonal path from G to H through N. Indeed, there exists $V \in \gamma \cap r_G$, hence γ between G and V can be shortened be $\overline{GV}$. Moreover, $\overline{NH}$ shortens the sub-path consisting in $\overline{GV}$ and in the sub-path of γ between V and H. In general, for any path γ there always exists a shorter polygonal path Δ, between $\tilde{E}$ and γ, tangent to $\tilde{E}$ in several points other than G and H. Applying Proposition 3, the thesis holds. ■

Proposition 5. *From any starting point A, any path γ of type $S^+ * E_2^+$ and $S^+ * E_1^-$ to B can be shortened by a path of type $S^+E_2^+$ or $E_2^+ * E_1^-$.*

Proof: Let $N \in SF(A)$ be the intersection point between extremal arc S^+ and extremal arc E_2^+. If arc E_2^+ intersects $\partial SF_2(A)$ in V, γ can be shortened between A and V by segment $\overline{AV}$ that is tangent to E_2^+ in V. Hence, path $S^+ * E_2^+$ is shortened by $S^+E_2^+$.

Let now consider a path of type $S^+ * E_1^-$ and, without loss of generality, consider the intersection point between S^+ and E_1^- be $N \in \partial SF_2(A)$. Indeed, if arc E_1^- intersects $\partial SF_2(A)$ in V, γ can be shortened between A and V by segment $\overline{AV}$. Considering now an arc E_2 passing through B, two cases occur:

- if arc E_2 intersects the first extremal arc S^+ in V_1, its also intersects $\partial SF_2(A)$. By using Proposition 4, arc E_2 shortens path $S^+ * E_1^-$ between V_1 and B. A path from A to B of type $S^+ * E_2^+$ has been obtained. By the considerations above, this path can be shortened by $S^+E_2^+$;
- otherwise, let us consider an arc E_2 through A. Let V its intersection point with E_1^-. By Proposition 4, the sub–path of γ between A and V can be shortened by E_2. Hence, a shorter path of type $E_2^+ * E_1^-$ is has been obtained.

■

Proposition 6. *From any starting point A, any path γ of type $S^+ * E_1^+$ or $S^+ * E_2^-$ to B can be shortened by a path of type $E_1^+S^+$ or $E_1^+ * E_2^-$.*

Proof: Let us consider first a path γ of type $S^+ * E_1^+$, and consider extremal arc E_1 for A where it is tangent to S^+. There always exists a point $V_1 \in \partial SB_2(B) \cap E_1$. Let V_2 be the intersection point between first arc S^+ of γ and straight line from V and B. The unfeasible straight line path from B to V_2 shortens path γ. By applying Proposition 3 to unfeasible polygonal path with vertices in V_1, V_2 and A, path $E_1^+S^+$ is shorter than a path $S^+ * E_1^+$.

For a path γ of type $S^+ * E_2^-$ from A to B with intersection point in V, without loss of generality $V \in \partial SF_1(A)$. Let us consider an extremal arc E_1 through B. Two cases can occur:

- if E_1 intersects first arc S^+ in V_1, by Proposition 4, $S^+ * E_2^-$ is longer than $S^+ * E_1^+$ through V_1 which, in turn, is longer than $E_1^+S^+$;
- otherwise the extremal arc E_1 from A to $V_2 \in E_1 \cap E_2^-$ shortens $S^+ * E_2^-$ from A to V_2. Hence, a path of type $E_1^+ * E_2^-$ shorter than $S^+ * E_2^-$ has been found.

■

Propositions 5 and 6 imply that paths of type $S^- * E_1^-$ and $S^- * E_2^-$ are not optimal. Indeed, they can be shortened by $S^-E_1^-$ and $E_2^-S^-$, respectively.

By using all previous results, a sufficient family of optimal paths is obtained in the following important theorem.

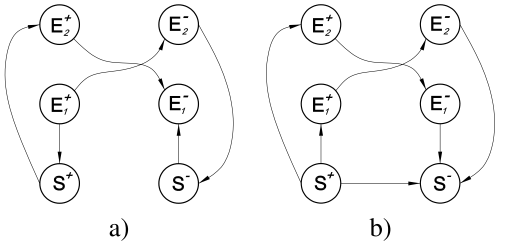


Fig. 6. Feasible extremals and sequence of extremals from point in D(P): a) in Side and Lateral cases ($\frac{\delta}{2} < \Gamma \leq \frac{\pi}{2}$). b) in Frontal case ($0 \leq \Gamma \leq \frac{\delta}{2}$).

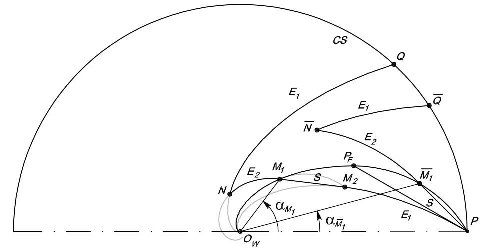


Fig. 7. Path of type $E_1^+ * E_2^- S^- E_1^-$ or the degenerate case of type $E_1^+ * E_2^- S^-$ from $Q \in CS$.

Theorem 4. *For $\frac{\delta}{2} < \Gamma \leq \frac{\pi}{2}$, i.e. Side and Lateral cases, and for any $Q \in D(P)$ to P there exists a shortest path of type $E_1^+ * E_2^- S^- E_1^-$ or of type $E_1^+ S^+ E_2^+ * E_1^-$. For $0 \leq \Gamma \leq \frac{\delta}{2}$, i.e. Frontal case, and for any $Q \in D(P)$ to P there exists a shortest path of type $S^+ E_1^+ * E_2^- S^-$ or of type $S^+ E_2^+ * E_1^- S^-$.*

Proof: According to all propositions above several concatenations of extremal have been proved to be non optimal. Considering extremals as node and, possibly optimal, concatenations of extremal as edges of a graph, the sufficient optimal languages $\mathscr{L}_O$ from Q in $D(P)$, for different values of Γ and δ, are described in fig. 6. Indeed, it is straightforward to observe that the number of switches between extremals is finite and less or equal to 3, for any value of Γ and δ. Hence, the thesis. ■

We now study the length of extremal paths from $C(P)$ to P in the sufficient family above. For space limitations, we proceed with the study of optimal path with $\frac{\delta}{2} < \Gamma \leq \frac{\pi - \delta}{2}$. However, the same analytical procedure can be used to solve the problem for any value of Γ.

Without loss of generality, it is sufficient to study the length of extremal paths of type $E_1^+ * E_2^- S^- E_1^-$ only from points Q on the semicircumference of $C(P)$ in the upper-half plane (denoted by CS). Indeed, up to a rotation, optimal paths of type $E_1^+ S^+ E_2^+ * E_1^-$ from the rest of $C(P)$ can be easily obtained. Referring to fig. 7, let the switching points of the optimal path be denoted by N, M_1 and M_2 or $\bar{N}$, $\bar{M}_1$ and $\bar{M}_2 \equiv P$, respectively, depending on the angular values α_{M_1} or $\alpha_{\bar{M}_1}$. Moreover, in order to do the analysis is useful to parameterize the family by the angular value $\alpha_{\bar{M}_1}$ of the switching point $\bar{M}_1$ along the arc $C_2(P)$ between P and Z or the angular value α_{M_1} of the switching point M_1 along the extremal E_1 between P_F and O_W.

Theorem 5. *For any point $Q \in CS$, the length of a path $\gamma \in \mathscr{P}_Q$ of type $E_1^+ * E_2^- S^- E_1^-$ is:*

- *for $0 \leq \alpha_{\bar{M}_1} \leq \phi_2 - \phi_1$, i.e. from P to Z (notice that the last arc has zero length):*

$$L = \rho_P \left\{ \frac{\cos \alpha_{M_1}}{\cos \phi_2} + \frac{1}{\cos \phi_1} + \right. \\ \left. - \frac{\cos \phi_1 + \cos \phi_2}{\cos \phi_1 \cos \phi_2} e^{\left(\psi_Q - \alpha_{M_1}\right) \frac{t_1 t_2}{t_2 - t_1}} \left(\frac{\sin\left(\phi_2 - \alpha_{M_1}\right)}{\sin \phi_2} \right)^{-\frac{t_1}{t_2 - t_1}} \right\}, \tag{7}$$

- *for $\alpha_{M_1} \geq \phi_2 - \phi_1$, i.e. from Z to O_W:*

$$L = \rho_P \left\{ \frac{2}{\cos \phi_1} + e^{-\alpha_{M_1} t_1} \left[\frac{\cos(\phi_2 - \phi_1)}{\cos \phi_2} - \frac{1}{\cos \phi_1} + \right. \right. \\ \left. \left. - \frac{\cos \phi_1 + \cos \phi_2}{\cos \phi_1 \cos \phi_2} e^{\left[\psi_Q - (\phi_2 - \phi_1)\right] \frac{t_1 t_2}{t_2 - t_1}} \left(\frac{\sin \phi_1}{\sin \phi_2} \right)^{-\frac{t_1}{t_2 - t_1}} \right] \right\}, \tag{8}$$

with $t_1 = 1/\tan \phi_1$ and $t_2 = 1/\tan \phi_2$.

The analytical expression for the length L is based on a direct computation. Having the path's length as a function of two parameters α_{M_1} or $\alpha_{\bar{M}_1}$ and ψ_Q, we are now in a position to minimize the length within the sufficient family.

Theorem 6. *Given a point $Q \in CS$,*

- *for $0 \leq \psi_Q \leq \psi_{R_1} := \frac{\sin(\phi_2 - \phi_1)}{\cos \phi_1 \cos \phi_2} \ln \left(\frac{\cos \phi_1 + \cos \phi_2}{\sin \phi_2 \sin(\phi_2 - \phi_1)} \right)$, optimal path is of type $E_1^+ * E_2^-$;*
- *for $\psi_{R_1} \leq \psi_Q \leq \psi_{R_2}$ with $\psi_{R_2} := (\phi_2 - \phi_1) + \psi_{R_1} + \tan \phi_2 \ln \left(\frac{\sin \phi_1}{\sin \phi_2} \right)$, optimal path is of type $E_1^+ * E_2^- S^-$;*
- *for $\psi_{R_2} \leq \psi_Q \leq \pi$ the optimal path is $E_1^+ * E_1^-$ through O_W.*

Moreover, for $\psi_Q = \psi_{R_2}$, any optimal path of type $E_1^+ * E_2^- S^- E_1^-$ turns out to have the same length ℓ of optimal path $E_1^+ * E_1^-$. Hence, for $\psi_Q = \psi_{R_2}$ also $E_1^+ * E_2^- S^- E_1^-$ is optimal.

Previous results have been obtained computing first and second derivatives of L and nonlinear minimization techniques.

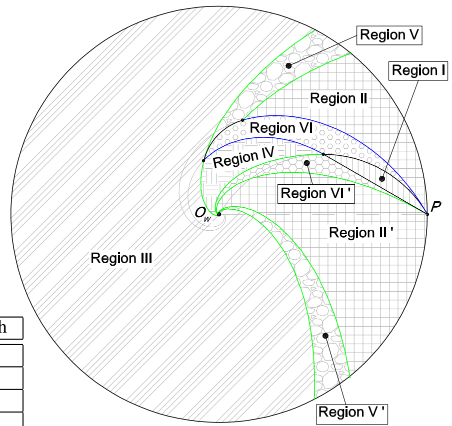
We are now interested in determining the locus of switching points between extremals in optimal paths.

Proposition 7. *For $Q \in CS$ with $0 < \psi_Q \leq \psi_{R_1}$, the switching locus is the arc of E_2 between P $M = (\rho_P \frac{\sin \phi_2 \sin(\phi_2 - \phi_1)}{\cos \phi_1 + \cos \phi_2}, \psi_M)$ (included), where $\psi_M = \tan \phi_2 \ln \left(\frac{\rho_P}{\rho_M} \right)$.*

Proof: From Theorem 6, the optimal path from $Q \in CS$ to P is of type $E_1^+ * E_2^-$. For $\psi_Q = \psi_{R_1}$ the intersection between E_1^+ and E_2^- is M. ■

Proposition 8. *For $Q \in CS$ with $\psi_{R_1} < \psi_Q < \psi_{R_2}$, the loci of switching points M_2 and N are the $\partial SF_2(P)$ and $\partial SF_2(M)$.*

Proof: For $Q \in CS$ with $\psi_{R_1} < \psi_Q < \psi_{R_2}$, considering the values of α_{M_2} obtained in the computations of Theorem 6 we obtain $M_2 \in \partial SF_2(P)$. Furthermore, substituting those values in the equation of the intersection point N between E_1 through Q and E_2 through M_2 we obtain $N \in \partial SF_2(M)$. ■



Region	Optimal Path
I	S^-
II	$E_1^+ * E_2^-$
II′	$E_2^+ * E_1^-$
III	$E_1^+ * E_1^-$
IV	$E_2^- S^- E_1^-$
V	$E_1^+ * E_2^- S^-$
V′	$S^+ E_2^+ * E_1^-$
VI	$S^- E_1^-$

Fig. 8. Optimal synthesis inside $D(P)$.

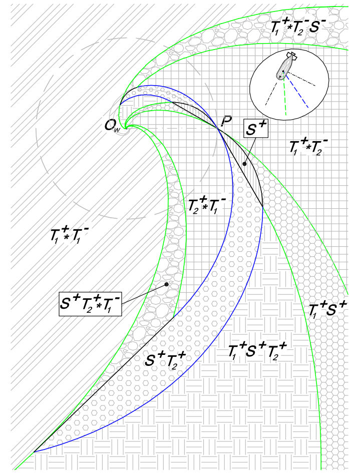


Fig. 9. Partition of the motion plane for $\frac{\delta}{2} < \Gamma < \frac{\pi-\delta}{2}$.

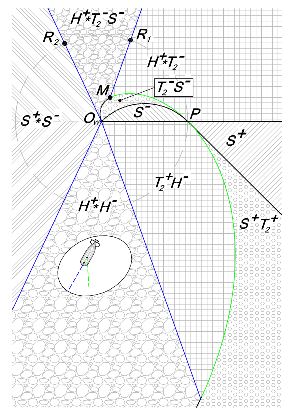


Fig. 10. Partition of the motion plane for $\Gamma = \delta/2$ (i.e. a SR border is aligned with the robot motion direction, Borderline Frontal).

Finally, for $Q \in CS$ with $\psi_{R_2} \leq \psi < \pi$, the switching locus reduces to the origin O_W since two extremal E_i intersect only in the origin for $i = 1, 2$.

V. Shortest paths from any point in the motion plane

The synthesis on $C(P)$ induce a partition in regions of $D(P)$. Indeed, for any $Q \in D(P)$, there exists a point $V \in C(P)$ such that the optimal path γ from V to P goes through Q. The Bellmann's optimality principle ensure the optimality of the sub–path from Q to P. Based on this construction the partition of $C(P)$ for $\frac{\delta}{2} < \Gamma \leq \frac{\pi-\delta}{2}$ (where $E_1 = T_1^R$ and $E_2 = T_2^R$) is reported in fig. 8.

A. Optimal paths for points outside $C(P)$

Function F_Q has been defined in 6 in order to transform paths starting from Q inside $C(P)$ in paths starting from $f_Q(P) = \left(\frac{\rho_P^2}{\rho_Q}, -\psi_Q\right)$ outside $C(P)$.

From other properties of F_Q, such as Proposition 1, we have also that an optimal path is mapped into an optimal path. Hence, the optimal synthesis from points outside $C(P)$ can be easily obtained mapping through map F_Q all borders of regions inside $C(P)$.

Proposition 9. *Given a border* $\mathbf{B}$ *and* $Q \in \mathbf{B}$ *map* F_Q *transforms:*

1) $\mathbf{B} = C(P)$ *into itself;*
2) $\mathbf{B} = \partial SF_2(Q)$ *in* $\partial SB_1(f_Q(P))$
3) $\mathbf{B} = \partial SF_1(Q)$ *in* $\partial SB_2(f_Q(P))$
4) $\mathbf{B} = E_i$ *in arcs of the same type* $(i = 1, 2)$

Proof: The proof of this proposition can be found in [12]. ■

Based on Proposition 9, the optimal synthesis of the entire motion plane in case of $\frac{\delta}{2} < \Gamma < \frac{\pi-\delta}{2}$ is reported in fig. 9.

B. Optimal synthesis in the Frontal case

We first obtain the synthesis of the Borderline Frontal case, i.e. $\Gamma = \frac{\delta}{2}$, reported in fig. 10 from the one obtained in the previous section.

Notice that, $E_1 = T_1^R$ of the Side case degenerates in a straight line H through O_W for $\Gamma = \frac{\delta}{2}$. Indeed, referring to fig. 8, points M_F and P_F degenerate on O_W. As a conseguenze, Region IV, IV and VI' while coordinates Ψ_{R_1} and Ψ_{R_2} of points R_1 and R_2 can be obtained from values in 6 replacing $\phi_1 = 0$.

In the Frontal case, $E_1 = H$ becomes a spiral T_1^L, straight lines from P and R_2 split in straight line and a spiral arc generating the partition reported in fig. 11. In this case, $\phi_1 < 0$ and points R_1 and R_2 do not lay on $C(P)$ but on a circumference through P with center $(0, -\rho_P \frac{\sin^2\phi_1 - \sin^2\phi_2}{2\sin\xi \sin\phi_1 \sin\phi_2})$, where $\xi = \frac{t_1+t_2}{t_1 t_2} \ln\left(\frac{\cos\phi_1 + \cos\phi_2}{\sin(\phi_2-\phi_1)}\right) + \frac{1}{t_1}\ln(-\sin\phi_1) - \frac{1}{t_2}\ln(\sin\phi_2)$. Notice that for $\phi_2 = -\phi_1$, this circumference coincide with $C(P)$ and the synthesis proposed in [12] is obtained.

Referring again to fig. 8, in the Borderline Side case ($\Gamma = \frac{\pi-\delta}{2}$, i.e. the SR border is aligned with the axle direction and $\phi_2 = \frac{\pi}{2}$), $E_2 = T_2^R$ degenerates in $E_2 = C$. Points $R_1 \equiv M$ and

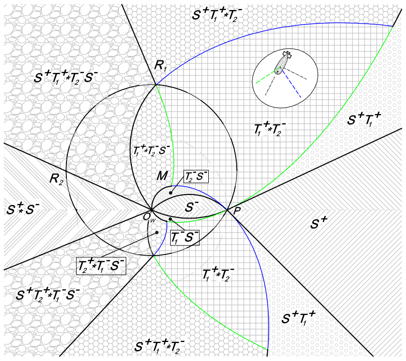


Fig. 11. Partition of the motion plane for $0 \leq \Gamma < \frac{\delta}{2}$, i.e. Frontal case.

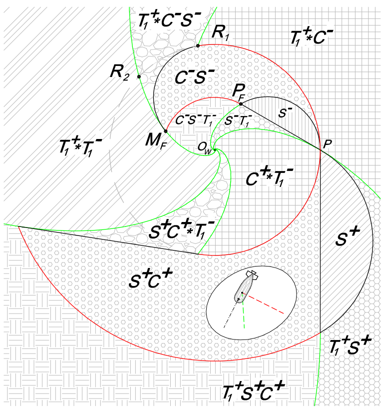


Fig. 12. Partition of the motion plane for $\Gamma = \frac{\pi-\delta}{2}$ (i.e. a SR border is aligned with the axle direction).

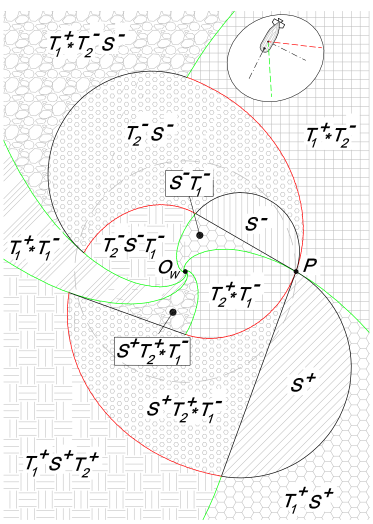


Fig. 13. Partition of the motion plane for $\frac{\pi-\delta}{2} \leq \Gamma < \frac{\pi}{2}$ (i.e. axle direction is included inside the SR).

R_2 lays on $C(P)$ with $\Psi_{R_1} = \frac{1+\sin\phi_1}{\cos\phi_1}$ and $\Psi_{R_2} = \frac{\pi}{2} - \phi_1 + \Psi_{R_1} + \tan\phi_1 \ln(\sin\phi_1)$. The obtained synthesis is reported in fig. 12. For the Lateral case $E_2 = C$ becomes $E_2 = T_2^L$ and the synthesis of the Lateral case, reported in fig. 13, can be obtained from the one in fig. 12.

The subdivision of the motion plane in case of $\frac{\pi}{2} < \Gamma \leq \pi$ can be easy obtained by using that one for $0 \leq \Gamma \leq \frac{\pi}{2}$ considering optimal path followed in reverse order, i.e. forward arc in backward arc and viceversa. Finally, a symmetry w.r.t. X_W axis of each subdivision of the motion plane for each $\Gamma \in [0, \pi]$ allows to obtain the correspond subdivision for $\Gamma \in [-\pi, 0]$.

VI. Conclusions and Future Work

A complete characterization of shortest paths for unicycle nonholonomic mobile robots equipped with a limited range side sensor systems has been proposed. A finite sufficient family of optimal paths has been determined based on geometrical properties of the considered problem. Finally, a complete shortest path synthesis to reach a point keeping a feature in sight has been provided. A possible extension of this work is to consider a bounded 3D SR pointing to any direction with respect to the direction of motion. A more challenging extension would be considering a different minimization problem such as the minimum time.

References

[1] N. Dias, C. Almeida, H. Ferreira, J. Almeida, A. Martins, A. Dias, and E. Silva, "Manoeuvre based mission control system for autonomos surface vehicle," *in Proceeedings of OCEANS '09*, May 2009.

[2] F. Langner, C. Knauer, W. Jans, and A. Ebert, "Side scan sonar image resolution and automatic object detection, classification and identification," *in Proceeedings of OCEANS '09*, May 2009.

[3] L. E. Dubins, "On curves of minimal length with a constraint on average curvature, and with prescribed initial and terminal positions and tangents," *American Journal of Mathematics*, pp. 457–516, 1957.

[4] X. Bui, P. Souères, J.-D. Boissonnat, and J.-P. Laumond, "Shortest path synthesis for Dubins non–holonomic robots," in *IEEE International Conference on Robotics and Automation*, 1994, pp. 2–7.

[5] J. A. Reeds and L. A. Shepp, "Optimal paths for a car that goes both forwards and backwards," *Pacific Journal of Mathematics*, pp. 367–393, 1990.

[6] H. Sussmann and G. Tang, "Shortest paths for the reeds-shepp car: A worked out example of the use of geometric techniques in nonlinear optimal controlò," Department of Mathematics, Rutgers University, Tech. Rep., 1991.

[7] H. Souères and J. P. Laumond, "Shortest paths synthesis for a car-like robot," *IEEE Transaction on Automatic Control*, pp. 672–688, 1996.

[8] H. Chitsaz, S. M. LaValle, D. J. Balkcom, and M. Mason, "Minimum wheel-rotation for differential-drive mobile robots," *The International Journal of Robotics Research*, pp. 66–80, 2009.

[9] H. Wang, Y. Chan, and P. Souères, "A geometric algorithm to compute time-optimal trajectories for a bidirectional steered robot," *IEEE Transaction on Robotics*, pp. –, 2009.

[10] D. Balkcom and M. Mason, "Time-optimal trajectories for an omnidirectional vehicle," *The International Journal of Robotics Research*, vol. 25, no. 10, pp. 985–999, 2006.

[11] M. Chyba, N. Leonard, and E. Sontag, "Optimality for underwater vehicle's," in *40th IEEE Conference on Decision and Control*, vol. 5, 2001, pp. 4204–4209.

[12] P. Salaris, D. Fontanelli, L. Pallottino, and A. Bicchi, "Shortest paths for a robot with nonholonomic and field-of-view constraints," *IEEE Transactions on Robotics*, vol. 26, no. 2, pp. 269–281, April 2010.

[13] ——, "Optimal paths for mobile robots with limited field of view camera," in *48th IEEE Conference on Decision and Control*, Dec 2009, pp. 8434–8439.

[14] S. Bhattacharya, R. Murrieta-Cid, and S. Hutchinson, "Optimal paths for landmark–based navigation by differential–drive vehicles with field–of–view constraints," *IEEE Transactions on Robotics*, vol. 23, no. 1, pp. 47–59, February 2007.

[15] A. Bryson and Y. Ho, *Applied optimal control*. Wiley New York, 1975.

[16] L. Cesari, *Optimization-theory and applications: problems with ordinary differential equations*. Springer-Verlag, New York, 1983.