Shortest paths for wheeled robots with limited Field-Of-View: introducing the vertical constraint

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***Abstract*— This paper presents the study of shortest paths for a robot with unicycle kinematics equipped with a limited Field–Of–View (FOV) camera, which must keep a given feature in sight. Previous works on this subject have provided the optimal synthesis for the case in which the FOV is limited in the left and right directions (H–FOV). Toward the final goal of obtaining the shortest paths synthesis for a realistic image plane modeled as a rectangle, in this paper we study the complementary case in which only upper and lower image plane direction are limited (V–FOV). A finite alphabet of extremal arcs is obtained. We also show that in some cases there exist no optimal path. However, we are always able to provide a path whose length approximates arbitrarily well any other shorter path.**

I. Introduction

The most important issues in mobile robotics, which deeply influence the accomplishing of assigned tasks and hence the control laws, concerns the directionality of motion (i.e. nonholonomic constraints) and limitation of sensory constraints (i.e. Field–Of–View (FOV)). Localization tasks or maintain visibility of some objects in the environment imply that some landmarks must be kept in sight. In visual servoing tasks this problem becomes particularly noticeable and in the literature several solutions have been proposed to overcome it. In [1] authors present a visual control approach consisting in a switching control scheme based on the epipolar geometry. To circumvent the limited FOV constraints, authors assume that the difference in depth from the initial position to the goal is greater than the side distance from the initial position to the goal, hence avoiding the need of high rotations. On the other hand, in [2] authors propose a visual control where the camera FOV constraints are alleviated because the algorithm works well with few landmarks. The constraint of the motion due to the limited FOV is also solved in [3] for a forklift vehicle by a control law derived using quadratic and barrier functions as a Lyapunov function candidate. In [4], the problem of controlling a leader–follower formation of two unicycle vehicle moving under visibility constraints in a known obstacle environment has been considered. Maintaining visibility is translated into controlling the robots so that system trajectories, starting from a visibility set, remain in it. The FOV problem has been also successfully solved for a unicycle–like vehicle also in [5], [6], [7] but, the resultant path is inefficient and not optimal.

"The research leading to these results has received funding from the European Union Seventh Framework Programme [FP7/2007-2013] under grant agreement n257462 HYCON2 Network of excellence".

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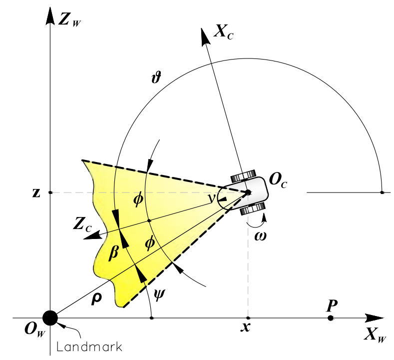
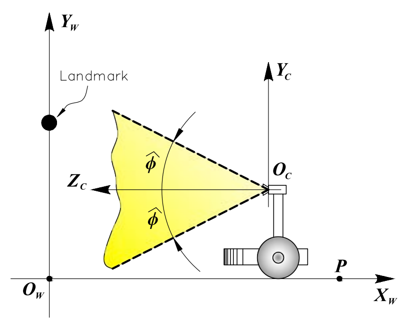


(a) Cartesian and polar coordinates of the robot and H-FOV constraints.



(b) V-FOV constraints.

Fig. 1. Mobile robot and systems coordinates. The robot's task is to reach P while keeping the landmark within a limited FOV (dashed lines).

In this paper we study the problem of maintain visibility of a fixed landmark for a unicycle vehicle equipped with a fixed camera with limited FOV. We consider a fixed camera instead of a pan-tilt mechanism for several reasons: first, the cost is three or five times the cost of a fixed camera, second a pan-tilt camera introduces a sensorized mechanical component which may be one of the more likely fail points. Moreover, it may introduce further estimation errors in case of the camera is used for localization porpoises and increase the complexity of the visual servo control.

The limited FOV problem is tackled here from an optimal point of view, i.e. finding shortest paths from any point on the motion plane to a goal position while keeping a given feature in sight during maneuvers. This problem has been already studied in [8] providing the shortest paths from any point on the motion plane to a desired final configuration. However, in [8] only right and left camera limits, i.e. the Horizontal–FOV (H–FOV) constraints, were taken into account, modeling the camera as a frontal and symmetric planar cone (i.e. the robot forward direction is included in the sensor cone and is the bisector of the apex angle), see figure 1(a). Moreover, in [9], based on the geometric properties of the synthesis proposed, optimal feedback control laws which are able to align the vehicle to the shortest path from the current configuration are defined for any point on the motion plane. In [10] a synthesis of shortest paths in case of lateral and side sensors (i.e. the robot forward direction is not included inside the planar cone), has been presented and includes, as a particular case, the synthesis provided in the earlier results [8].

The model adopted in previous papers does not consider the upper and lower limits of the camera: indeed, the vehicle can approach and reach the feature position maintaining it in sight. The final goal is to obtain the shortest paths synthesis for the realistic case of FOVs modeled as a four–sided right rectangular pyramid (see figure 2), hence having right, left, upper and lower limits. In this paper another step toward the final goal is done studying the complementary case in which only upper and lower camera limits, i.e. the Vertical–FOV (V–FOV) constraints, are considered as in figure 1(b). The impracticality of paths that reach a compact set around the feature and the loss of geometrical properties of optimal arcs, lead to a substantially more complex analysis for the definition of the sufficient family of optimal paths with respect to previous works. Indeed, in the V–FOV case we will obtain a finite alphabet of optimal arcs. We also show that is some cases there exist no optimal path. However, we are always able to provide a path whose length approximates arbitrarily well any other shorter path.

II. Problem Definition

Consider a vehicle moving on a plane where a right-handed reference frame $\langle W \rangle$ is defined with origin in O_w and axes X_w, Z_w. The configuration of the vehicle is described by $\xi(t) = (x(t), z(t), \theta(t))$, where $(x(t), z(t))$ is the position in $\langle W \rangle$ of a reference point in the vehicle, and $\theta(t)$ is the vehicle heading with respect to the X_w axis (see figure 1). We assume that the dynamics of the vehicle are negligible, and that the forward and angular velocities, $v(t)$ and $\omega(t)$ respectively, are the control inputs of the kinematic model of the vehicle. Choosing polar coordinates (see figure 1), the kinematic model of the unicycle-like robot is

$$\begin{bmatrix} \dot{\rho} \\ \dot{\psi} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} -\cos\beta & 0 \\ \frac{\sin\beta}{\rho} & 0 \\ \frac{\sin\beta}{\rho} & -1 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}. \quad (1)$$

We consider vehicles with bounded velocities which can turn on the spot. In other words, we assume

$$(v, \omega) \in U, \quad (2)$$

with U a compact and convex subset of $\mathbb{R}^2$, containing the origin in its interior.

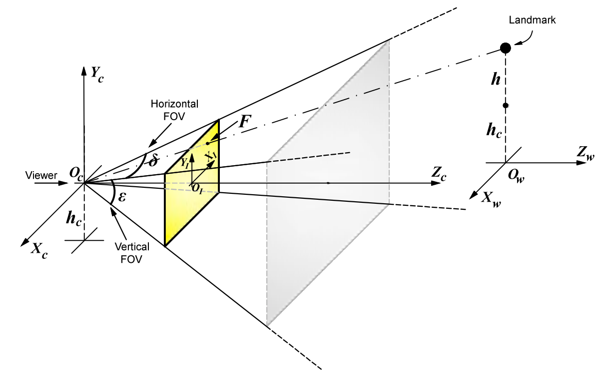


Fig. 2. Sensor model: four-sided right rectangular pyramid with $\varepsilon = 2\hat{\phi}$ and $\delta = 2\phi$, where $\hat{\phi}$ is half of the V–FOV angular aperture, whereas ϕ is half of the H–FOV angular aperture.

The vehicle is equipped with a rigidly fixed pinhole camera with a reference frame $\langle C \rangle = \{O_c, X_c, Y_c, Z_c\}$ such that the optical center O_c corresponds to the robot's center $[x(t), z(t)]^T$ and the optical axis Z_c is aligned with the robot's forward direction. Cameras can be generically modeled as a four-sided right rectangular pyramid, as shown in figure 2. We will refer to those angles as the vertical and horizontal angular aperture of the sensor, respectively. Moreover, $\hat{\phi}$ is half of the V–FOV angular aperture, whereas ϕ is half of the H–FOV angular aperture.

We assume that the feature to be kept within the on-board limited FOV sensor is placed on the axis through the origin O_w, perpendicular to the plane of motion and height $h + h_c$ from it (see figure 2), so that its projection on the motion plane coincides with the center O_w (see figure 1). Moreover, let us consider the position of the robot target point P to lay on the X_w axis, with coordinates $(\rho, \psi) = (\rho_P, 0)$. In order to maintain the feature within the limited FOV sensor, following inequality constraints must be anytime satisfied during robot's maneuvers:

$$\phi \pm \beta \geq 0, \quad (3)$$

$$\rho \cos\beta \geq \frac{h}{\tan\hat{\phi}} = R_b, \quad (4)$$

where inequalities (3) concern H-FOV limits, whereas inequality (4) concerns V-FOV limits.

In [8], authors have provided a complete characterization of shortest paths towards a goal point taking into account only constraints (3) (i.e. H-FOV constraints) and hence modeling the camera FOV as a planar cone moving with the robot. Optimal paths consist of at most 5 arcs of three types: rotations on the spot (denoted by the symbol $*$), straight lines (S) and left and right logarithmic spirals (T^L and T^R). The obtained result is a subdivision of the motion plane in region such that an optimal sequence of symbols (corresponding to an optimal path) is univocally associated to a region and completely describe the shortest path from each point in that region to P.

In this paper, we take into account the V-FOV constraint given by equation (4) neglecting the H-FOV ones by considering an horizontal FOV aperture $\phi = \frac{\pi}{2}$ and the most interesting case in which $\hat{\phi}$ is less than $\pi/2$. The goal is to determine, for any point $Q \in \mathbb{R}^2$ in the robot space, the shortest path from Q to P, such that the feature F is maintained in the FOV of the sensor. In other words, the objective is to minimize the length of the path covered by the center of the vehicle, i.e. to minimize the cost functional

$$L = \int_0^\tau |v|\, dt\,, \tag{5}$$

under the *feasibility constraints* (2), (1) and (4), respectively. Here τ is the time needed to reach P, that is $\rho(\tau) = \rho_P$ and $\psi(\tau) = 0$. Notice that, cost functional (5) does not weigh β, i.e. rotations on the spot have zero length. As a consequence, in the following these maneuvers, denoted by $*$, will be used only to properly connect other maneuvers.

III. Analysis of V–FOV Constraints

In this section, we study the effect of the V-FOV constraint (4) on the motion plane and we characterize the paths followed by the vehicle while activating the constraint (i.e. the constraint is verified as an equality).

Definition 1: Let $Z_0 = \{(\rho, \psi) | \rho < R_b\}$ the circle centered in the origin with radius R_b. Let $Z_1 = (X_w \times Y_w) \setminus Z_0$.

Remark 1: Z_0 is the set of points in $\mathbb{R}^2$, in polar coordinates, that violates the V–FOV constraint (4). Notice that points with $\rho = R_b$ verify the constraint with $\beta = 0$.

The V-FOV constraint is activated for those configurations with

$$\rho \cos \beta = R_b. \tag{6}$$

It is possible to determine the relationship between the control inputs v and ω required to follow a path along which (6) is verified. With simple computations, it can be verified that the relation between ψ and β

$$\psi = \psi_b + \tan \beta - \beta \tag{7}$$

holds along the path. Paths characterized by equation (6) and (7) are curves known as *involutes of a circle* expressed by polar coordinates where ψ_b is the angular coordinate of a point on the involute such that $\beta = 0$, and hence $\rho = R_b$. The involute of a circle is the path traced out by a point on a straight line that rolls around a circle without slipping (see figure 3). Moreover, for any point on circumference C_{R_b} with radius R_b and centered in O_w there are two involutes of circle one rotating clockwise with $\beta > 0$ and another counterclockwise with $\beta < 0$. We refer to these two involutes as *Left* and *Right*, and by symbols I^L and I^R, respectively. The adjectives "Left" and "Right" indicate the half–plane where the involute starts for an on-board observer aiming at the landmark.

Following the same Hamiltonian-based approach used in [8], the extremal arcs (i.e. curves that satisfy necessary conditions for optimality, see [11]) are the involutes I^R and I^L, the turn on the spot $*$ and the straight lines S. Moreover, as extremal arcs can be executed by the vehicle in either forward or backward direction, superscripts $+$ and $-$ will be used in the following in order to make this explicit. As a consequence, extremal paths consist of sequences, or *words*, comprised of symbols in the finite alphabet $\mathscr{A} = \{*, S^+, S^-, I^{R+}, I^{R-}, I^{L+}, I^{L-}\}$. The set of possible words generated by the symbols in $\mathscr{A}$ is a language $\mathscr{L}$.

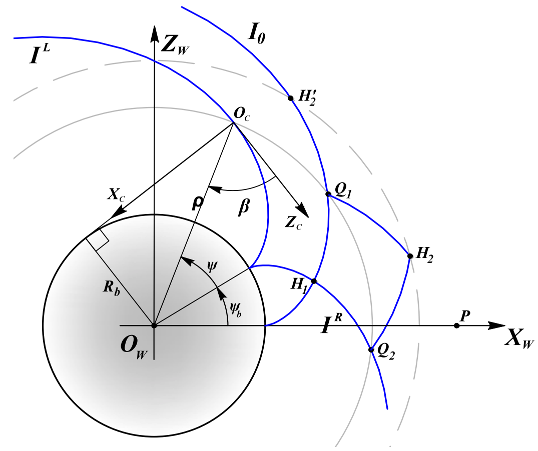


Fig. 3. Extremal arcs for the V-FOV: the involutes of a circle of radius R_b. I_0 is the involute with $\psi_b = 0$. Path $\mathscr{C}_1 = I^{L+} * I^{R-}$ and path $\mathscr{C}_2 = I^{R-} * I^{L+}$ both between Q_1 and Q_2.

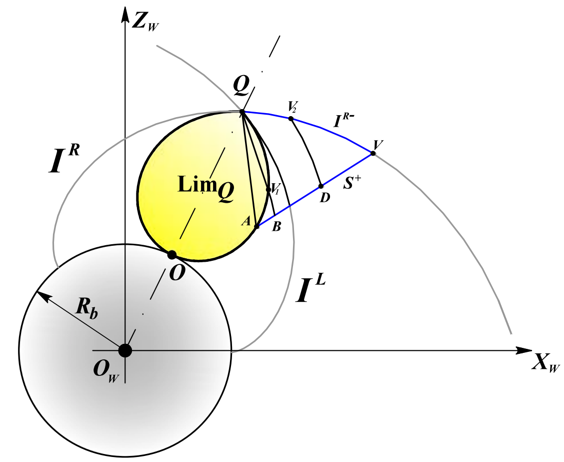


Fig. 4. Construction used in the proof of Proposition 7.

A. Properties and lengths of Involute curve

In order to determine the best concatenation of extremals we first need to better characterize the properties of the involute. First notice that involutes are invariant w.r.t. rotations (around O_w) and axial symmetry (with the axis through O_w). Given such invariance properties, in the computations that follow, we will consider points on the left involute with $\psi_b = 0$ denoted by I_0.

Remark 2: Consider a point Q at distance ρ_Q from the origin, the point lays on I_0 if $\psi_Q = \tan \beta_Q - \beta_Q$ and $\beta_Q = \arccos\left(\frac{R_b}{\rho_Q}\right)$, i.e. β_Q is the solution of (6) with $\rho = \rho_Q$. Equivalently, a point Q with angle ψ_Q lays on I_0 if $\rho_Q = \frac{R_b}{\cos \beta_Q}$ where β_Q is solution of $\Psi(\beta_Q) = \psi_Q$ where

$$\Psi(\beta) = \tan \beta - \beta. \tag{8}$$

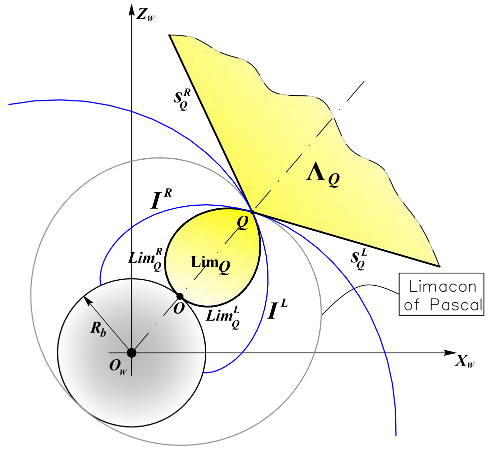


Fig. 5. Region Lim_Q with its border $\partial Lim_Q = Lim_Q^R \cup Lim_Q^L$ and cone Λ_Q delimited by half–lines s_Q^R and s_Q^L.

Notice that this function is invertible for $\beta \in [0, \frac{\pi}{2})$. The inverse will be denoted by $\Psi^{-1}(\psi)$.

Remark 3: The function Ψ is increasing and convex for $\beta \in [0, \frac{\pi}{2})$, i.e. $\Psi'(\beta) > 0$, $\Psi''(\beta) > 0$ $\forall \beta \in [0, \pi/2)$ where Ψ' and Ψ'' are the first and second order derivatives of function Ψ. As a consequence, the inverse function $\Psi^{-1}(\psi)$ is increasing and concave.

We are now able to compute lengths of involute arcs. Given a point $Q = (\rho_Q, \psi_Q) \in I_0$, the length of the involute arc from Q to $Q_b = (R_b, 0)$ is $\ell_0(Q) = \frac{R_b}{2\cos^2(\beta_Q)}$.

Given two points Q_1 and Q_2 on I_0 with $\rho_{Q_1} > \rho_{Q_2}$ the length of the involute arc between the points is

$$\ell(Q_2, Q_1) = \ell_0(\beta_{Q_2}) - \ell_0(\beta_{Q_1}). \tag{9}$$

where β_i and ψ_i are computed as in remark 2.

IV. Optimal concatenation of extremals

Let $\mathscr{P}_Q$ be the set of all feasible extremal paths from Q to P. In the rest of the paper we will study which combination of extremals belong to the optimal path in $\mathscr{P}_Q$.

Due to the symmetry of the problem, the analysis of optimal paths in $\mathscr{P}_Q$ can be done considering only the upper half plane w.r.t. the X_w axis.

Definition 2: An extremal path in $\mathscr{P}_Q$ (i.e. from Q to P), described by a word $w \in \mathscr{L}$ is a *palindrome symmetric path* if the word is palindrome and the path is symmetric w.r.t. the bisectrix of angle $\widehat{QO_WP}$.

Proposition 1: For any path in $\mathscr{P}_Q$ with $\rho_Q = \rho_P$ there always exists a palindrome symmetric path in $\mathscr{P}_Q$ whose length is shorter or equal.

Proof: The proof is the same of the equivalent proposition for the H-FOV case and reported in [8]. Indeed, extremals I^R (as it occurs for logarithmic spirals) are transformed into I^L (and viceversa) with a symmetry with respect to a straight line through the origin. ∎

We first consider extremals of type S and we study the set of points from which P is reachable through extremals S^+ or S^-.

Definition 3: For a point $Q \in \mathbb{R}^2$, let Lim_Q^R (Lim_Q^L) denote the arc of the Limaçon [12] from Q to O such that, $\forall V \in Lim_Q^R$ (Lim_Q^L), $\widehat{QVO_w} = \pi - \bar{\beta}$, with $\bar{\beta} = \arctan\left(\frac{\rho_Q}{Rb}\sin\beta\right)$, in the half-plane on the right (left) of $\overline{QO}_w$ (cf. figure 5). Also, let Lim_Q denote the region delimited by Lim_Q^R and Lim_Q^L from Q to O.
We will refer to Lim_Q^R (Lim_Q^L) as the right (left) $\hat{\phi}$-arc in Q.

Definition 4: For a point $Q \in \mathbb{R}^2$, let s_Q^R (s_Q^L) denote the half-line from Q forming an angle $\psi_Q + \tilde{\beta}$ ($\psi_Q - \tilde{\beta}$), where $\tilde{\beta} = \arccos\left(\frac{h}{\rho_Q \tan\hat{\phi}}\right)$, with the X_w axis (cf. figure 5). Also, let Λ_Q denote the cone delimited by s_Q^R and s_Q^L.
We will refer to s_Q^R (s_Q^L) as the right (left) $\hat{\phi}$-radius in Q.

Proposition 2: For any starting point Q, all points of Lim_Q (Λ_Q) are reachable by a forward (backward) straight path without violating the V–FOV constraints.
For space limitations the proof of the Proposition is omitted[1].

Theorem 1: Given two arbitrary points Q and P, one of the following conditions is verified.

1) There exists a shortest path of type $S^+ I^{L+} * I^{R-} S^-$.
2) The infimum of the cost functional L is not reached and hence the shortest path does not exist.

To prove Theorem 1, we establish first a few preliminary results.

Proposition 3: If an optimal path in $\mathscr{P}_Q$ includes a segment of type S^+ with extremes in A, B, then either $B = P \in Lim_A$ or $B \in Lim_A^R \cup Lim_A^L$.

Proof: The proof can be done as for the H-FOV case by substituting the arc of circle C_A with the arc of Limaçon Lim_A, see [8]. ∎

Remark 4: The argument of Proposition 3 can be repeated for any point A' on the S^+ segment ending in B. Hence, for any forward segment $\overline{AB}$ of an optimal path $\gamma \in \mathscr{P}_Q$, it holds either $B \in \bigcap_{A' \in \overline{AB}} \partial Lim_{A'}^R$ or $B \in \bigcap_{A' \in \overline{AB}} \partial Lim_{A'}^L$. Notice that this holds also for the particular case $B = P$.

Proposition 4: If an optimal path $\gamma \in \mathscr{P}_Q$ includes a segment of type S^- with extremes in B, A, then either $A = P \in \Lambda_B$ or $A \in s_G^R \cup s_G^L$.

Proof: The proof can be done as for the H-FOV case by substituting the half-lines r_G with half-lines s_G, see [8]. ∎

Proposition 5: If a path $\gamma(t)$ is optimal, then its angle $\psi(t)$ is monotonic.
The proof is in [8].

Remark 5: By applying Proposition 5 to optimal paths from Q in the upper half–plane to P, and noticing that $\psi_Q \geq \psi_P = 0$, the angle is non increasing. Hence optimal paths in the upper half–plane do not include counter–clockwise arcs I^{R+} and I^{L-}.

Proposition 6: Any path of type $S^- * I^{R-}$ (resp., $I^{L+} * S^+$) can be shortened by a path of type $I^{R-} S^-$ (resp., $S^+ I^{L+}$).

Proof: Let A and B be the initial and final points of the $S^- * I^{R-}$, and let A_1 be the switching point between S^- and I^{R-} (see fig. 6). Without loss of generality, we assume that

[1]The proof can be found at `http://www.centropiaggio.unipi.it/sites/default/files/HFOV_TR13_0.pdf`.

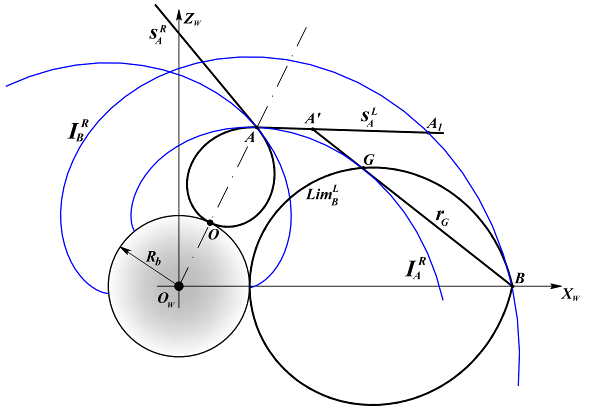


Fig. 6. Construction used in the proof of Proposition 6.

A_1 belongs to s_A^L, the left ϕ–radius in A (if not, the path can be shortened by a path of the same type for which this is true). Let G be the intersection point between the involute I_A^R through A and the ϕ–arc Lim_B^R through B. By Definitions 3,4, and the properties of involutes, the line s_G through B and G is tangent to I_A^R in G, while s_A^L is tangent to I_A^R in A. Let A' be the intersection of s_G with s_A^L. The segment $\overline{A'B}$ is shorter than the sub–path $S^- * I^{R-}$ from A' to B through A_1. By properties of convex curves, the feasible involute arc I_A^R from A to G shortens $\overline{AA'} \cup \overline{A'G}$, hence the thesis. The proof for $I^{L+}S^+$ is analogous. ■

Proposition 7: Any path of type $I^{R-} * S^+$ (resp., $S^- * I^{L+}$) can be shortened by a path of type $I^{R-} * I^{L+}$, $S^+ I^{L+}$ or S^+ (resp., $I^{L-} * I^{R+}$, $I^{L-} S^-$ or S^-).

Proof: Referring to figure 4, consider first a path of type $I^{R-} * S^+$ from a point Q with a switching point in V. If the final point of S^+ falls into Lim_Q or on its border (such as point A in the figure) the path $I^{R-} * S^+$ can be shortened by S^+ from Q to A directly. If the final point is B (i.e. below I^L through Q) we consider I^L through Q that intersects Lim_Q in V_1 and a shorter path is $S^+ I^L$ from Q to B through V_1. If the final point is D (i.e. below I^R and above I^L through Q), we consider I^L through D that intersects I^R through Q in V_2 and a shorter path is $I^L * I^R$ from Q to D through V_1. The thesis is true also for $S^- * I^{L+}$ since the path is symmetric to $I^{R-} * S^+$ with respect to a line through the origin. ■

To conclude the analysis of optimal extremal concatenations we need to study concatenations of type $I^{L+} * I^{R-}$ and $I^{R-} * I^{L+}$. For this purpose we first need to study the properties of involutes.

V. Infinite sequences of involute arcs

In this section we will show that the particular characteristics of the involute arcs may give raise to a minimum (and finite) arc length consisting of infinite involutes of infinitesimal length.

Proposition 8: Consider $Q_1 = (\rho_{Q_1}, \psi_{Q_1})$ and $Q_2 = (\rho_{Q_2}, \psi_{Q_2})$ with $\rho_{Q_1} = \rho_{Q_2}$ and $\psi_{Q_1} > \psi_{Q_2}$. The points Q_1 and Q_2 can be connected by two paths, each one symmetric with respect to the bisectrix of angle $\widehat{Q_1 O_w Q_2}$, consisting of two pairs of involute curves $\mathscr{C}_1 = I^{L+} * I^{R-}$ and $\mathscr{C}_2 = I^{R-} * I^{L+}$. Let $H_1 = (\rho_{H_1}, \psi_{H_1})$ and $H_2 = (\rho_{H_2}, \psi_{H_2})$ be the points of intersection of the involute curves on $\mathscr{C}_1$ and $\mathscr{C}_2$ respectively, i.e. $\rho_{H_1} < \rho_{Q_i} < \rho_{H_2}$ and $\psi_{H_1} = \psi_{H_2}$.

Denoting by $L(\mathscr{C}_1)$ and $L(\mathscr{C}_2)$ the length of the curves $\mathscr{C}_1$ and $\mathscr{C}_2$, it holds

1) $\rho_{Q_1} \geq \rho_{max} \Rightarrow L(\mathscr{C}_1) \leq L(\mathscr{C}_2)\ \forall \rho_{H_1}$.
2) $\rho_{Q_1} \in (\rho_{min}, \rho_{max}), \rho_{H_1} \geq \rho_{min} \Rightarrow L(\mathscr{C}_1) < L(\mathscr{C}_2)$
3) $\rho_{Q_1} \leq \rho_{min} \Rightarrow L(\mathscr{C}_1) \geq L(\mathscr{C}_2)\ \forall \rho_{H_1}$.

where $\rho_{min} = \sqrt{2} R_b$, $\rho_{max} = R_b \sqrt{1 + \tan^2 \beta_0}$ and β_0 is associated to a point $Q_0 = (\rho_0, \psi_0) \in I_0$ such that $\ell(Q_0, Q_b) = \ell(Q_0', Q_0)$ where $Q_b = (R_b, 0)$ and $Q_0' = (\rho_0', 2\psi_0) \in I_0$.

For space limitations the proof of the Proposition is omitted, see note 1.

Corollary 1: Consider $\rho > \rho_{min} = \sqrt{2} R_b$, the point $Q_1 = (\rho, \psi_{Q_1}) \in I_0$ and a point $Q_2 = (\rho, \psi_{Q_2})$ with $\psi_{Q_1} > \psi_{Q_2}$, if

$$\psi_{Q_1} - \Psi(\pi/4) \geq \frac{\psi_{Q_1} - \psi_{Q_2}}{2} \tag{10}$$

then the optimal trajectory consisting of involutes from Q_1 to Q_2 is $\mathscr{C}_1 = I^{L+} * I^{R-}$.

Proof: From Remark 2, $\Psi(\pi/4)$ is the angle of $Q_{\sqrt{2}} = (R_b \sqrt{2}, \Psi(\pi/4)) \in I_0$. Condition (10) ensures that the angle $\widehat{Q_1 O_w Q_{\sqrt{2}}} > \widehat{Q_1 O_w H_1}$ and hence $\rho_{H_1} \geq \sqrt{2} R_b$. By applying the conditions 1 and 2 of Proposition 8 we have that $L(\mathscr{C}_1) < L(\mathscr{C}_2)$. Using this result for any subpath of $\mathscr{C}_1$, symmetric with respect to the line from O_w through H_1, the same conditions of Proposition 8 hold proving that $\mathscr{C}_1$ can not be shortened with other involutes concatenations. ■

Corollary 2: Consider $\rho < \rho_{min} = \sqrt{2} R_b$, the point $Q_1 = (\rho, \psi_{Q_1}) \in I_0$ and a point $Q_2 = (\rho, \psi_{Q_2})$ with $\psi_{Q_1} > \psi_{Q_2}$, if

$$\Psi(\pi/4) - \psi_{Q_1} \geq \frac{\psi_{Q_1} - \psi_{Q_2}}{2} \tag{11}$$

then the optimal trajectory consisting of involutes from Q_1 to Q_2 is $\mathscr{C}_2 = I^{R-} * I^{L+}$.

Proof: Similarly to previous corollary proof, condition (11) ensures that the angle $\widehat{Q_{\sqrt{2}} O_w Q_1} > \widehat{Q_1 O_w H_1}$ (recall that H_1 and H_2 are aligned with O_w) and hence $\rho_{H_2} \leq \sqrt{2} R_b$. By applying the conditions 3 of Proposition 8 we have that $L(\mathscr{C}_2) < L(\mathscr{C}_1)$. Using this result for any subpath of $\mathscr{C}_2$, symmetric with respect to the line from O_w through H_2, the same conditions of Proposition 8 hold proving that $\mathscr{C}_2$ can not be shortened with other involutes concatenations. ■

Proposition 9: Consider $\rho = \rho_{min} = \sqrt{2} R_b$ and $Q_1 = (\rho, \psi_{Q_1}) \in I_0$, for any $Q_2 = (\rho, \psi_{Q_2})$ with $\psi_{Q_2} < \psi_{Q_1}$ it holds $L(\mathscr{C}_2) < L(\mathscr{C}_1)$. Moreover, the shortest path between Q_1 and Q_2 does not exists.

Proof: The first statement is a straightforward consequence of the condition 3 of Proposition 8. Moreover, by applying Corollary 1 to $\mathscr{C}_2$ this path can be shortened with the path, consisting of 2 identical pairs of paths of type $\mathscr{C}_2$, symmetric with respect to the bisectrix of angle $\widehat{Q_1 O_w Q_2}$. This reasoning can be iterated to the two new obtained paths and so on concluding that any path of this kind can be shortened increasing the number of switching points on the circumference of radius $\sqrt{2}R_b$. Hence, the optimal path does not exist. ■

Based on Propositions 8 and 9 we can deduce the following

Theorem 2: Consider a point $Q_1 = (\rho_{Q_1}, \psi_{Q_1}) \in I_0$ and a point $Q_2 = (\rho_{Q_2}, \psi_{Q_2})$ with $\rho_{Q_1} = \rho_{Q_2}$ and $\psi_{Q_1} > \psi_{Q_2}, \forall \psi_{Q_2}$. If

1) $\rho_{Q_1} \geq \rho_{max} \Rightarrow \mathscr{C}_1$ is the shortest path consisting of involutes $\forall \rho_{H_1}$.
2) $\rho_{Q_1} \leq \rho_{min} \Rightarrow \mathscr{C}_2$ is the shortest path consisting of involutes $\forall \rho_{H_1}$.
3) $\rho_{Q_1} \in (\rho_{min}, \rho_{max})$
   a) $\rho_{H_1} \geq \rho_{min} \Rightarrow \mathscr{C}_1$ is the shortest path consisting of involutes.
   b) $\rho_{H_1} \leq \rho_{min} \Rightarrow$ the shortest path does not exist.

For those cases in which the optimal path does not exist, we are now interested in the infimum of the lengths of the paths consisting of infinite involutes.

Theorem 3: Consider $\rho = \rho_{min} = \sqrt{2}R_b$, the point $Q_1 = (\rho, \psi_{Q_1}) \in I_0$ and a point $Q_2 = (\rho, \psi_{Q_2})$ with $\psi_{Q_1} > \psi_{Q_2}$ and $\psi_{Q_1} - \psi_{Q_2} \leq \pi$. The infimum length of a path consisting of infinite subpaths of type $\mathscr{C}_2$ from Q_1 to Q_2 is finite and

$$L_{inf}(Q_1, Q_2) = \sqrt{2}(R_b\sqrt{2}(\psi_{Q_1} - \psi_{Q_2}))$$

i.e. $\sqrt{2}$ times the length of the circular arc from Q_1 to Q_2 on the circumference with radius $R_b\sqrt{2}$.

Proof: Let us define $\mathscr{C}_2^{(0)}$ as a path of type $\mathscr{C}_2$ connecting Q_1 to Q_2 and β_0 be the (positive) heading angle of the vertex H (corresponding to intersection between the two involutes of $\mathscr{C}_2^{(0)}$). The length of the path is given by $L(\mathscr{C}_2^{(0)}) = 2(\ell_0(\beta_0) - \ell_0(\pi/4))$. On the other hand, by Proposition 9, a shorter curve can be found considering a path which consists of 2 identical pairs of paths of type $\mathscr{C}_2$. Denoting by $\mathscr{C}_2^{(1)}$ the new curve associated to this partition, its length is

$$L(\mathscr{C}_2^{(1)}) = 4\left[\ell_0\left(\Psi^{-1}\left(\frac{\Psi(\beta_0) - \Psi(\pi/4)}{2} + \Psi(\pi/4)\right)\right) + \right. \\ \left. -\ell_0(\pi/4)\right].$$

The procedure can be iterated obtaining a generic path $\mathscr{C}_2^{(n)}$ consisting of n subpaths of type $\mathscr{C}_2$ and the total length $L_n := L(\mathscr{C}_2^{(n)})$ is

$$L_n = 2^{n+1}\left[\ell_0\left(\Psi^{-1}\left(\frac{\Psi(\beta_0) - \Psi(\pi/4)}{2^n} + \Psi(\pi/4)\right)\right) + \right. \\ \left. -\ell_0(\pi/4)\right].$$

The minimal length $L_{inf}(Q_1, Q_2)$ satisfies $L_{inf}(Q_1, Q_2) \leq \lim_{n\to\infty} L_n$. We can compute the limit by using Taylor expansion. Let us recall that

$$\ell_0(\pi/4 + s) = \ell_0(\pi/4) + 2R_b s + 4R_b s^2 + o(s^2); \tag{12}$$

moreover, by the inverse function derivative rule

$$\Psi^{-1}(\Psi(\pi/4) + s) = \pi/4 + s - 2s^2 + o(s^2). \tag{13}$$

In this way, naming $\zeta := \Psi(\beta_0) - \Psi(\pi/4)$ and replacing $\ell_0(\cdot)$ and $\Psi^{-1}(\cdot)$ with (12) and (13), we have

$$\begin{aligned}\lim_{n\to\infty} L_n &= \lim_{n\to\infty} 2^{n+1}\left(\ell_0\left(\pi/4 + \frac{\zeta}{2^n} - \frac{2\zeta^2}{2^{2n}} + o\left(\frac{1}{2^{2n}}\right)\right) + \right. \\ &\quad \left. -\ell_0(\pi/4)\right) = \lim_{n\to\infty} 2^{n+1}\left(\frac{2R_b\zeta}{2^n} + o\left(\frac{1}{2^{2n}}\right)\right) \\ &= 4R_b\zeta = 2R_b(\Psi(\beta_0) - \Psi(\pi/4)).\end{aligned}$$

Recall that $\beta_0 = \Psi^{-1}\left(\frac{(\psi_{Q_1} - \psi_{Q_2})}{2} + \Psi(\pi/4)\right)$, than

$$\zeta = \Psi\left(\Psi^{-1}\left(\frac{(\psi_{Q_1} - \psi_{Q_2})}{2} + \Psi(\pi/4)\right)\right) - \Psi(\pi/4) = \\ = \frac{(\psi_{Q_1} - \psi_{Q_2})}{2}.$$

As a consequence, $L_{inf}(Q_1, Q_2) \leq 4R_b\zeta = 2R_b(\psi_{Q_1} - \psi_{Q_2})$, i.e. $\sqrt{2}$ times the length of the circumference arc from Q_1 to Q_2.

On the other hand, for any $\xi \in [0, \infty)$ and for a path of type $\mathscr{C}_2$ among two points, the following holds

$$2\left(\ell_0\left(\Psi^{-1}\left(\xi + \Psi(\pi/4)\right)\right) - \ell_0(\pi/4)\right) \geq 4R_b\xi. \tag{14}$$

This can be proved as follows. Define the auxiliary function

$$p(\sigma) = \Psi(\pi/4 + \sigma) - \Psi(\pi/4), \quad \sigma \in [0, \pi/4);$$

it can be easily verified that

$$p(0) = 0, \quad \lim_{\sigma\to\pi/4} p(\sigma) = \infty, \quad p'(\sigma) > 0 \quad \forall \sigma \in [0, \infty).$$

As a consequence for any fixed $\xi \in [0, \infty)$ there exists a unique $\sigma \in [0, \pi/4)$ such that $\xi = p(\sigma)$. Let us suppose now that for some $\xi_0 \in [0, \infty)$ condition (14) does not hold, i.e.

$$\ell_0\left(\Psi^{-1}\left(\xi_0 + \Psi(\pi/4)\right)\right) - \ell_0(\pi/4) < 4R_b\xi_0, \tag{15}$$

or equivalently for $\xi_0 = p(\sigma_0)$

$$\ell_0(\pi/4 + \sigma_0) - \ell_0(\pi/4) < 4R_b\Psi(\pi/4 + \sigma_0) - 4R_b\Psi(\pi/4). \tag{16}$$

Defining $\chi(s) = \ell_0(s) - 4R_b\Psi(s)$, inequality (16) leads to

$$\chi(\pi/4 + \sigma_0) < \chi(\pi/4). \tag{17}$$

On the other hand the function $\chi(s)$ verifies

$$\chi'(s) = 2R_b\left(\frac{\tan s}{\cos^2 s} - \frac{2}{\cos^2 s} + 2\right) \geq 0 \quad \forall s \in [\pi/4, \pi/2):$$

this fact contradicts (17) and hence (15) is not achievable and hence inequality (14) is proved.

As a consequence, for $\xi = \frac{\psi_{Q_1} - \psi_{Q_2}}{2}$ condition (14) implies that the length of an infinite number of $\mathscr{C}_2$ path connecting two arbitrary points Q_1 and Q_2 on the circumference with radius $R_b\sqrt{2}$ is greater than $\sqrt{2}\,\overset{\frown}{Q_1Q_2}$ where $\overset{\frown}{Q_1Q_2}$ is the length of the circular arc between Q_1 and Q_2. Hence

$$L_{inf}(Q_1, Q_2) \geq \sqrt{2}\,\overset{\frown}{Q_1Q_2}.$$

In conclusion it has been shown that

$$\sqrt{2}\,\overset{\frown}{Q_1Q_2} \leq L_{inf}(Q_1, Q_2) \leq \sqrt{2}\,\overset{\frown}{Q_1Q_2},$$

i.e. the length of a path consisting of infinite subpath of type $\mathscr{C}_2$ is finite. ■

Even if theorem 3 proves that the length of a path of infinite subpaths of type $\mathscr{C}_2$ is finite, from a robotic point of view this leads to an impracticable path. However, depending on the accuracy of motors which move the wheels of the robot, this type of path can be approximated by a finite sequence of involutes with an error as smaller as more accurate is the motor. The following remark gives the minimum number n of "jump" of type $\mathscr{C}_2$ on circumference of radius $R_b\sqrt{2}$ such that the length of any other shorter paths is no longer than an arbitrarily small $\varepsilon > 0$.

Remark 6: Given a trajectory $C^{(n)}$ from Q_1 to Q_2 on the circumference of radius $Rb\sqrt{2}$ consisting of n identical subpaths of type $\mathscr{C}_2$ and a positive parameter $\varepsilon > 0$, we are interested in finding the minimum value $n = n(\varepsilon)$ such that

$$L(C^{(n)}) - L_{inf}(Q_1, Q_2) \leq \varepsilon.$$

Since $\Psi(\pi/4 + s) \geq \Psi(\pi/4) + s \;\forall s \in \left[0, \frac{\pi}{4}\right)$, the following inequality for the inverse function can be deduced

$$\Psi^{-1}(\Psi(\pi/4) + s) \leq \pi/4 + s \quad \forall s \in \left[0, \frac{\pi}{4}\right).$$

As a consequence

$$L(C^{(n)}) = 2n\left(\ell_0\left(\Psi^{-1}\left(\frac{\zeta}{n} + \Psi(\pi/4)\right)\right) - \ell_0(\pi/4)\right) \leq$$
$$\leq 2n\left(\ell_0\left(\pi/4 + \frac{\zeta}{2n}\right) - \ell_0(\pi/4)\right).$$

Assuming $s \in [0, 0.3]$, the following estimate holds

$$\ell_0(\pi/4 + s) \leq \ell_0(\pi/4) + 2R_b s + c_0 s^2, \tag{18}$$

where $c_0 := \ell_0(\pi/4 + 1) - 3R_b$; since we are interested in finding a good approximation of the shortest path, it is reasonable to consider small increments of the variable ζ and in particular it can be assumed $\zeta \leq 0.3$. Substituting (18) in $L(C^{(n)})$, we obtain

$$L(C^{(n)}) \leq 2R_b\zeta + c_0\frac{\zeta^2}{n} \quad \forall n \geq 2,$$

or equivalently

$$L(C^{(n)}) - L_{inf}(Q_1, Q_2) \leq c_0\frac{\zeta^2}{n} \quad \forall n \geq 2.$$

As a consequence the bound $L(C^{(n)}) - L_{inf}(Q_1, Q_2) \leq \varepsilon$ is ensured if

$$n \geq \frac{c_0(\psi_{Q_1} - \psi_{Q_2})^2}{2\varepsilon}.$$

We are finally able to prove Theorem 1

Proof: **[Proof of Theorem 1.]** According to Propositions 6–7, Remark 5 and Proposition 9, the number of switches between extremals may be infinite. However, for those paths that do not intersect the circumference of radius $\sqrt{2}R_b$ such switches are actually at most 3. Indeed, in such cases $I^{L+} * I^{R-} * I^{L+} * I^{R-}$ is always longer than $I^{L+} * I^{R-}$ and the optimal path is of type $S^+ I^{L+} * I^{R-} S^-$.

On the other hand, for those paths that intersect the circumference of radius $\sqrt{2}R_b$ an ε-sub-optimal path with a finite number $n(\varepsilon)$ of switching can be obtained as shown in Remark 6. ■

VI. Conclusions and Future Works

Based on the peculiarities of the involute arcs, optimal paths of finite length consisting in an infinite number of involutes arcs has been obtained. Hence while the optimal alphabet is finite, as in the H-FOV, in some cases the optimal path does not exit. For those cases, the computation of the minimum number of switches between involutes to guarantee that the length of the finite switching path is not larger than ε with respect any other shorter path has been provided.

Future works will be devoted to further study the optimal paths for the V-FOV case. In particular, an optimal synthesis of the motion plane is still missing, i.e. a global partition of the plane induced by the shortest paths, such that a word in the optimal language is univocally associated to a region and completely describes the shortest path from any starting point in that region to the goal point.

References

[1] G. López-Nicolás, C. Sagüés, J. Guerrero, D. Kragic, and P. Jensfelt, "Switching visual control based on epipoles for mobile robots," *Robotics and Autonomous Systems*, vol. 56, no. 7, pp. 592 – 603, 2008.

[2] X. Zhang, Y. Fang, and X. Liu, "Visual servoing of nonholonomic mobile robots based on a new motion estimation technique," in *Proceedings of the 48th IEEE Conference on Decision and Control (CDC)*, 2009, pp. 8428 –8433.

[3] A. Widyotriatmo, G.-Y. Hong, and K.-S. Hong, "Configuration control of a forklift vehicle using vision system with limited angle of view," in *International Conference on Control Automation and Systems (ICCAS)*, 2010, pp. 76–81.

[4] D. Panagou and V. Kumar, "Maintaining visibility for leader-follower formations in obstacle environments," in *IEEE International Conference on Robotics and Automation (ICRA)*, 2012, pp. 1811–1816.

[5] P. Murrieri, D. Fontanelli, and A. Bicchi, "A hybrid-control approach to the parking problem of a wheeled vehicle using limited view-angle visual feedback," *Int. Jour. of Robotics Research*, vol. 23, no. 4–5, pp. 437–448, April–May 2004.

[6] N. Gans and S. Hutchinson, "Stable visual servoing through hybrid switched system control," *IEEE Transactions on Robotics*, vol. 23, no. 3, pp. 530–540, June 2007.

[7] ——, "A stable vision-based control scheme for nonholonomic vehicles to keep a landmark in the field of view," in *IEEE International Conference on Robotics and Automation (ICRA)*, 2007, pp. 2196 – 2201.

[8] P. Salaris, D. Fontanelli, L. Pallottino, and A. Bicchi, "Shortest paths for a robot with nonholonomic and field-of-view constraints," *IEEE Transactions on Robotics*, vol. 26, no. 2, pp. 269–281, April 2010.

[9] P. Salaris, L. Pallottino, S. Hutchinson, and A. Bicchi, "From optimal planning to visual servoing with limited fov," in *Intelligent Robots and Systems (IROS), IEEE/RSJ International Conference on*, 2011, pp. 2817–2824.

[10] P. Salaris, L. Pallottino, and A. Bicchi, "Shortest paths for finned, winged, legged, and wheeled vehicles with side-looking sensors," *Int. Jour. of Robotics Research*, vol. 31, no. 8, pp. 997–1017, 2012.

[11] A. Bryson and Y. Ho, *Applied optimal control*. Wiley New York, 1975.

[12] E. H. Lockwood, *"The Limaçon." Ch. 5 in A Book of Curves*. Cambridge, England: Cambridge University Press, 1967, pp. 44–51.