ON THE OBSERVABILITY OF MOBILE VEHICLE LOCALIZATION

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In this paper, we consider the problem of localizing a mobile vehicle moving in an unstructured environment, based on triangulation measurements derived from processed optical information. The problem is shown to be intrinsically nonlinear, in the sense that the linear approximation of the system has different structural properties than the original model. In particular, linearized approximations are non–observable, while results obtained from differential–geometric nonlinear system theory prove the possibility of reconstructing the position and orientation of the vehicle and the position of the obstacles in the environment from optical information.

1 Introduction

One of the main technical difficulties in applying mobile robots to unstructured environments is the problem of localization of the vehicle with respect to the environment, and of constructing a map of the environment itself. The problem is important for instance for a rover exploring an unknown terrain, such as was the case for the Sojourner explorer in the recent NASA mission of Pathfinder on Mars. On the other hand, many everiday applications on Earth call for solutions to the same problem. There is in fact a consolidated trend in industrial AGV systems to move from traditional wire–guided systems to optically guided systems, using laser or camera heads on the vehicle to locate it on the factory floor. The advantage of the latter techniques is apparent, in terms of drastically reducing the cost and rigidity of fixed nets of active or passive magnetic devices placed under the floor, and allowing more variate trajectories to be executed by the AGV's. On the other hand, the technology of optical localization is rather new, and several problems are still encountered, related to both its technological aspects and to methodology to be used in filtering and merging data from different sources. A good treatment for these problems is that of Borenstein [2], and references therein.

In this paper, we deal with the problem of localization and map building for a mobile vehicle endowed with odometric and optical sensors (laser or camera heads). In section 2 we recast the problem as one in nonlinear observability, and results obtained from differential–geometric nonlinear system theory are compared with those resulting from a linearized model, showing how the problem is intrinsically nonlinear. In section 3 we discuss the use of an Extended Kalman Filter for the system under consideration.

2 Observability

We consider a system comprised of a mobile vehicle, such as a robotic rover in a planetary exploration mission, which moves in an unknown environment with the aim of localizing itself and the environment features (in the rover case, e.g., rocks and geological formations). The vehicle is endowed with a sensor head such as a radial laser rangefinder or movable camera, whose data are assumed here to have been preprocessed so as to yield a measurement of the azimuth angle in the horizontal plane between the line joining the obstacle features with the head position, and the direction of movement of the vehicle (or any other direction fixed w.r.t. the vehicle). An information on distance of the target from the head is not considered to be available, due to the fact that such measurement is hard to obtain accurately from current laser or camera sensors.

Both the vehicle initial position and orientation, and the obstacle positions, are unknown (or, more generally, known up to some apriori probability distribution). The task is to reconstruct such information from angular measurements.

A model of the system that captures most salient features of the problem, yet lends itself to simple analytical results, is used, which is based on the following assumptions: the vehicle moves on a plane, and object features are represented as points of the plane. Among the features that the sensor head detects in the robot environment, we will distinguish between those belonging to objects with unknown positions (which we shall call *targets*), and those belonging to objects whose absolute position is known, which will be referred to as *markers*.

The vehicle is a kinematic unicycle (this is the case of NASA's Sojourner, for instance), whose dynamics are slow enough to be neglected (dynamics do not add much to the problem structure, while considerably increase formal complexity). The mathematical model is described as:

$$\begin{bmatrix} \dot{\xi} \\ \dot{\zeta} \\ \dot{\theta} \\ \dot{\xi}_1 \\ \dot{\zeta}_1 \\ \vdots \\ \dot{\xi}_N \\ \dot{\zeta}_N \end{bmatrix} = \begin{bmatrix} \cos(\xi) \\ \sin(\zeta) \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} u_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} u_2, \tag{1}$$

where ξ, ζ represent the coordinates of the position of the vehicle with respect to some arbitrary fixed reference frame, and θ is the angle between the ξ axis and the direction of motion of the rover; ξ_i, ζ_i are the position coordinates (in the same reference) of the i–th target; u_1 is the vehicle forward velocity, while u_2 is its angular velocity. Observe that the system is in the standard form of nonlinear systems which are linear in control, i.e.

$$\dot{\mathbf{x}} = \mathbf{g}_1(\mathbf{x})u_1 + \mathbf{g}_2(\mathbf{x})u_2.$$

The measurement process is modelled by N equations for target observations in the form

$$y_i = h_i(\mathbf{x}) = \pi + \mathrm{atan2}\frac{\zeta - \zeta_i}{\xi - \xi_i} - \theta, \quad i = 1, \ldots, N, \tag{2}$$

and by M further measurements relating to markers, whose absolute position $(\bar{\xi}_i, \bar{\zeta}_i)$ is known, as

$$y_i = h_i(\mathbf{x}) = \pi + \mathrm{atan2}\frac{\zeta - \bar{\zeta}_i}{\xi - \bar{\xi}_i} - \theta, \quad i = N+1, \ldots, N+M. \tag{3}$$

The localization problem is therefore stated, in terms of the above model, as a problem of reconstructing the initial state $\mathbf{x}(0)$ of the system (1) from measurement of the $M+N$ output angles $\mathbf{y}(t)$. The odometry information is simply embodied in the fact that, in reconstructing the state of the sytem, information on input velocities can be used explicitely (i.e., in our kinematic model inputs $u_1(t), u_2(t)$ coincide with odometric measurements, up to measurement errors).

The linear approximation of the system (about an arbitrary initial state $\hat{\mathbf{x}}$ and zero control) is given by the triplet $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ as follows:

$$\mathbf{A} = \mathbf{0}_{(n+3)\times(n+3)}; \quad \mathbf{B} = \left[\mathbf{g}_1(\hat{\mathbf{x}}) \; \mathbf{g}_2(\hat{\mathbf{x}}) \right];$$

$$\mathbf{C} = \left[\begin{array}{ccc|cccccc} c_{1,1} & c_{1,2} & -1 & -c_{1,1} & -c_{1,2} & 0 \; 0 & \cdots & 0 & 0 \\ \vdots & \vdots & -1 & \vdots & \vdots & \vdots \; \vdots & \cdots & \vdots & \vdots \\ c_{N,1} & c_{N,2} & -1 & 0 & 0 & 0 \; 0 & \cdots & -c_{N,1} & -c_{N,2} \\ \hline c_{N+1,1} & c_{N+1,2} & -1 & & & & & & \\ \vdots & \vdots & -1 & & & \mathbf{0}_{M\times N} & & & \\ c_{N+M,1} & c_{N+M,2} & -1 & & & & & & \end{array} \right],$$

where

$$c_{i,1} = \frac{\partial h_i}{\partial \xi} = -\frac{\hat{\zeta} - \hat{\zeta}_i}{d_i^2} \quad i = 1, \ldots, N$$

$$c_{i,2} = \frac{\partial h_i}{\partial \zeta} = \frac{\hat{\xi} - \hat{\xi}_i}{d_i^2} \quad i = 1, \ldots, N$$

$$c_{i,1} = \frac{\partial h_i}{\partial \xi} = -\frac{\hat{\zeta} - \bar{\zeta}_i}{d_i^2} \quad i = N+1, \ldots, M+N$$

$$c_{i,2} = \frac{\partial h_i}{\partial \zeta} = \frac{\hat{\xi} - \bar{\xi}_i}{d_i^2} \quad i = N+1, \ldots, M+N$$

and $d_i^2 = (\hat{\xi} - \hat{\xi}_i)^2 + (\hat{\zeta} - \hat{\zeta}_i)^2$, $i = 1, \ldots, N$; $d_i^2 = (\hat{\xi} - \bar{\xi}_i)^2 + (\hat{\zeta} - \bar{\zeta}_i)^2$, $i = N+1, \ldots, N+M$. Because of the absence of the drift term, the observability matrix of the linearized system is $\mathbf{C} \in \mathbb{R}^{(M+N)\times(2N+3)}$, and $\mathsf{rank}\,(\mathbf{C}) = N + \min(3, M+N)$ in generic configurations (i.e., assuming that no pair of markers and/or targets are aligned with the head center). The unobservable subspace has dimension N if $M+N \geq 3$, or $3-M$ if $M+N \leq 3$. Hence, the linearized system is always unobservable if there are targets ($N \neq 0$), and can only be observable if there are at least 3 markers ($M \geq 3$). This results contrast with intuition, and with the common practice in navigation and surveying of making the point by triangulation, where the problem is solved with $M = 2, N = 0$.

The observability analysis of the full nonlinear model solves this apparent contradiction. Observe first that, if no markers are available, observations from the vehicle will at most allow localization up to a rigid motion of the plane where the vehicle and the targets lie; and that, if one marker only is available, localization can be done at most up to a rigid rotation of the plane about the marker position. In the lack of an absolute reference (as would be the case of a rover on a planet's ground), we will assume that a reference frame is fixed arbitrarily, by choosing a fixed point as the origin and a fixed direction as the ζ axis of the reference frame. This is tantamount to considering one of the targets as a marker in the origin, and another target as a second marker aligned on the ζ axis (see fig.1). We will show in the following that two markers are enough for completely reconstructing the state of the system from measurements[a]. Observability

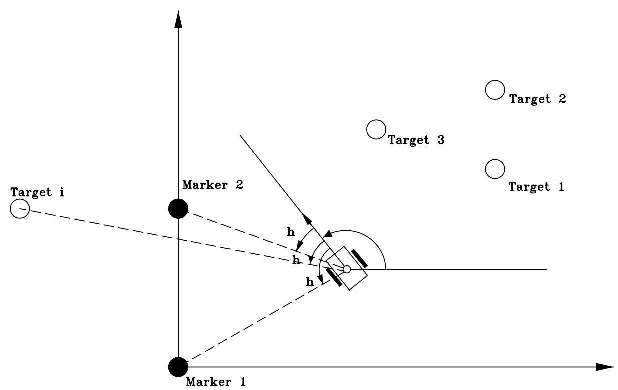

Figure 1: A mobile robot in an unknown environment with markers and targets

of system (1)–(2)–(3) can be checked (see e.g. the book of Isidori [4]) by computing the dimension of the smallest codistribution that contains the output one–forms, and is invariant with the control vector fields, in symbols $< \mathbf{g}_1, \mathbf{g}_2 |$ span $\{dh_1, dh_2, \ldots, dh_{N+M}\} >$. Consider that

$$dh_1 = \left[-\frac{\zeta-\zeta_1}{d_1^2} \; \frac{\xi-\xi_1}{d_1^2} \; -1 \; \frac{\zeta-\zeta_1}{d_1^2} \; -\frac{\xi-\xi_1}{d_1^2} \; 0 \; 0 \cdots 0 \; 0 \right],$$

$$\vdots$$

$$dh_2 = \left[-\frac{\zeta-\zeta_2}{d_2^2} \; \frac{\xi-\xi_2}{d_2^2} \; -1 \; 0 \; 0 \; \frac{\zeta-\zeta_2}{d_2^2} \; -\frac{\xi-\xi_2}{d_2^2} \cdots 0 \; 0 \right],$$

$$\vdots$$

$$dh_N = \left[-\frac{\zeta-\zeta_N}{d_N^2} \; \frac{\xi-\xi_N}{d_N^2} \; -1 \; 0 \; 0 \cdots \frac{\zeta-\zeta_N}{d_N^2} \; -\frac{\xi-\xi_N}{d_N^2} \right],$$

[a] In fact, there is no need to assume that the distance a between the two targets used as markers is known, for such an information can be reconstructed from outputs as well. However, this assumption will be made for the sake of reducing computations.

$$dh_{N+1} = \left[-\frac{\zeta}{d_{N+1}^2} \; \frac{\xi}{d_{N+1}^2} \; -1 \; 0 \; 0 \; 0 \; 0 \cdots 0 \; 0 \right],$$
$$dh_{N+2} = \left[-\frac{\zeta-a}{d_{N+2}^2} \; \frac{\xi}{d_{N+2}^2} \; -1 \; 0 \; 0 \; 0 \; 0 \cdots 0 \; 0 \right]$$

where $d_i^2 = (\xi-\xi_i)^2 + (\zeta-\zeta_i)^2$, $i = 1, \ldots, N$; $d_{N+1}^2 = \xi^2 + \zeta^2$, $d_{N+2}^2 = \xi^2 + (\zeta-a)^2$. Furthermore, the Lie derivatives of these covector fields along the control vector fields are easily computed as

$$L_{\mathbf{g}_2} dh_i = 0, \quad i = 1, \ldots, N+2,$$

and

$$L_{\mathbf{g}_1} dh_i = \left[\frac{p_i S_\theta + q_i C_\theta}{d_i^4} \; \frac{p_i C_\theta - q_i S_\theta}{d_i^4} \; \frac{(\zeta-\zeta_i)S_\theta + (\xi-\xi_i)C_\theta}{d_i^2} \; 0 \; 0 \cdots \right.$$
$$\left. \cdots \; \underset{\text{4+2(i-1)-th} \downarrow \text{position}}{-\frac{p_i S_\theta + q_i C_\theta}{d_i^4}} \; -\frac{p_i C_\theta - q_i S_\theta}{d_i^4} \cdots 0 \; 0 \right],$$

for $i = 1, \ldots, N$, where $p_i = (\zeta-\zeta_i)^2 - (\xi-\xi_i)^2$, $q_i = 2(\xi-\xi_i)(\zeta-\zeta_i)$, $S_\theta = \sin_\theta$, and $C_\theta = \cos\theta$. Also, we have

$$L_{\mathbf{g}_1} dh_{N+1} =$$
$$= \left[\frac{p_{N+1} S_\theta + q_{N+1} C_\theta}{d_{N+1}^4} \; \frac{p_{N+1} C_\theta - q_{N+1} S_\theta}{d_{N+1}^4} \; \frac{\zeta S_\theta + \xi C_\theta}{d_{N+1}^2} \; \mathbf{0}_{1\times 2N} \right],$$

and

$$L_{\mathbf{g}_1} dh_{N+2} =$$
$$= \left[\frac{p_{N+2} S_\theta + q_{N+2} C_\theta}{d_{N+2}^4} \; \frac{p_{N+2} C_\theta - q_{N+2} S_\theta}{d_{N+2}^4} \; \frac{(\zeta-a) S_\theta + \xi C_\theta}{d_{N+2}^2} \; \mathbf{0}_{1\times 2N} \right],$$

where $p_{N+1} = \zeta^2 - \xi^2$, $q_{N+1} = 2\xi\zeta$, $p_{N+2} = (\zeta-a)^2 - \xi^2$, $q_{N+2} = 2\xi(\zeta-a)$. Finally, compute

$$L_{\mathbf{g}_2} L_{\mathbf{g}_1} dh_{N+1} =$$
$$= \left[-\frac{p_{N+1} C_\theta + q_{N+1} S_\theta}{d_{N+1}^4} \; \frac{p_{N+1} C_\theta - q_{N+1} S_\theta}{d_{N+1}^4} \; \frac{\zeta C_\theta - \xi S_\theta}{d_{N+1}^2} \; \mathbf{0}_{1\times 2N} \right],$$

and

$$L_{\mathbf{g}_2} L_{\mathbf{g}_1} dh_i =$$
$$= \left[\frac{p_{N+1} C_\theta - q_{N+1} S_\theta}{d_{N+1}^4} \; \frac{p_{N+1} S_\theta - q_{N+1} C_\theta}{d_{N+1}^4} \; \frac{(\zeta-\zeta_i) C_\theta - (\xi-\xi_i) S_\theta}{d_{N+1}^2} \right.$$
$$\left. 0 \quad 0 \quad \cdots \underset{\text{4+2(i-1)-th} \downarrow \text{position}}{-\frac{p_i C_\theta + q_i S_\theta}{d_i^4}} \; \frac{-p_i S_\theta + q_i C_\theta}{d_i^4} \cdots 0 \; 0 \right],$$

Consider first the observability of the robot position and orientation, i.e. the self–localization problem. Denote $\Omega^{<1-3>}$ the first three columns of the matrix

$$\Omega = \begin{bmatrix} dh_1 \\ \vdots \\ dh_{N+2} \\ L_{\mathbf{g}_1} dh_1 \\ L_{\mathbf{g}_1} dh_2 \\ L_{\mathbf{g}_2} L_{\mathbf{g}_1} dh_1 \end{bmatrix}.$$

In order for these columns to be independent, hence for the localization of the vehicle to be distinguishable, it suffices that either of the minors

$$\det\begin{bmatrix} dh_{N+1}^{<1-3>} \\ dh_{N+2}^{<1-3>} \\ L_{\mathbf{g}_1} dh_{N+1}^{<1-3>} \end{bmatrix} = \frac{a(\zeta-a)(\zeta C_\theta - \xi S_t heta)}{d_{N+2}^2 d_{N+1}^4};$$

$$\det\begin{bmatrix} dh_{N+1}^{<1-3>} \\ dh_{N+2}^{<1-3>} \\ L_{\mathbf{g}_1} dh_{N+2}^{<1-3>} \end{bmatrix} = \frac{a\zeta((\zeta-a)C_\theta - \xi S_t heta))}{d_{N+2}^2 d_{N+1}^4};$$

$$\det\begin{bmatrix} dh_{N+1}^{<1-3>} \\ dh_{N+2}^{<1-3>} \\ L_{\mathbf{g}_2} L_{\mathbf{g}_1} dh_{N+1}^{<1-3>} \end{bmatrix} = -\frac{a(\zeta-a)(\xi C_\theta + \zeta S_t heta)}{d_{N+2}^2 d_{N+1}^4};$$

is non-zero. It can be easily checked that all these minors simultaneously vanish only in the case $a = 0$, which means that the two markers coincide[b].

In order to assess the possibility of reconstructing the position of the $i - th$ target, it suffices to study the two minors

$$\det\begin{bmatrix} dh_i^{<4+2(i-1)-5+2(i-1)>} \\ L_{\mathbf{g}_1} dh_i^{<4+2(i-1)-5+2(i-1)>} \end{bmatrix};$$

$$\det\begin{bmatrix} dh_i^{<4+2(i-1)-5+2(i-1)>} \\ L_{\mathbf{g}_2} L_{\mathbf{g}_1} dh_i^{<4+2(i-1)-5+2(i-1)>} \end{bmatrix},$$

which never vanish simultaneously. Complete observability of the location of the vehicle and of targets with two markers follows.

It may be of interest to notice that, if the vehicle could only move along straight lines (i.e., $\mathbf{g}_2 = 0$), observability of the vehicle's states and of the targets would still hold, except for the cases where the vehicle moves along the line joining the markers, and along a line pointing to some target, respectively (see fig.2, fig.3). If the vehicle could only rotate, observability would be lost of both vehicle and target positions (see fig.4).

3 EKF for localization

In this section we discuss the problem of building a filter for the localization problem, or, in other words, a filter that uses instantaneous knowledge of inputs and outputs

[b] since it can be verified that span $\Omega^{<1-3>}$ has dimension 2 in this case, there exist a 1–dimensional submanifold of indistinguishable locations, amounting to rotations of the robot about the origin

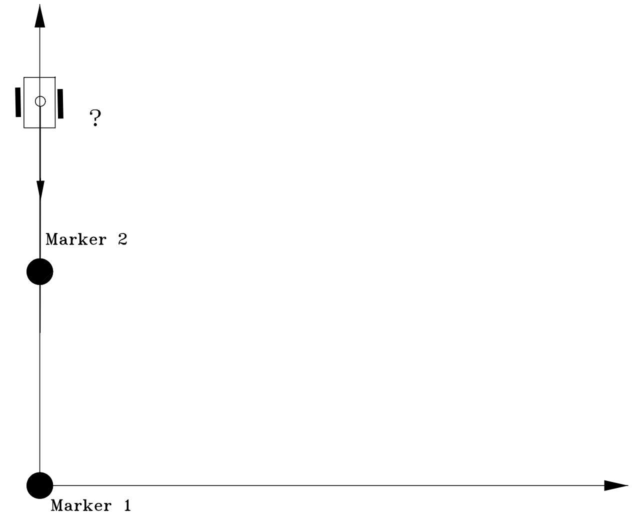

Figure 2: A vehicle that moves along straight lines cannot localize itself if only if the trajectory aims at the two markers.

to estimate the state of system (1)–(2)–(3). This problem is notoriously difficult, and a large number of papers have been devoted to proposing different schemes for solving it. The most common way of approaching the problem is to use an Extended Kalman Filter. Several authors underscored that EKF–based treatment of localization data is often troublesome. Our direct experience in laboratory with application of EKF to localization problems has shown that the filter convergence properties are very mcuh prone to initialization of filter parameters (measurement and process covariances, e.g.), and is inclined to unpredictably diverge at some points during exploration.

Our previous discussion in section 1 showed that the linear approximation of the system is not observable. Since the EKF uses at each step the linear approximation of the system to observe its state, there are reasons to be dubious about success of applying the EKF to the localization problem.

To illustrate this, consider the EKF equations in continuous time (as derived e.g. by Gelb [3]) for estimating the state $\mathbf{x} \in \mathbb{R}^n$ of a general nonlinear system (affine in control). Let the system equations be

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x},t) + \mathbf{G}(\mathbf{x},t)\mathbf{u}(t) + \mathbf{q}(t), \\ \mathbf{y} &= \qquad\qquad \mathbf{h}(\mathbf{x},t) + \mathbf{r}(t) \end{aligned}$$

where $\mathbf{u} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^m$, and $\mathbf{q} \in \mathbb{R}^n, \mathbf{q} \sim N(0, \mathbf{Q}(t))$, $\mathbf{r} \in \mathbb{R}^m, \mathbf{r} \sim N(0, \mathbf{R}(t))$ are zero–mean uncorrelated

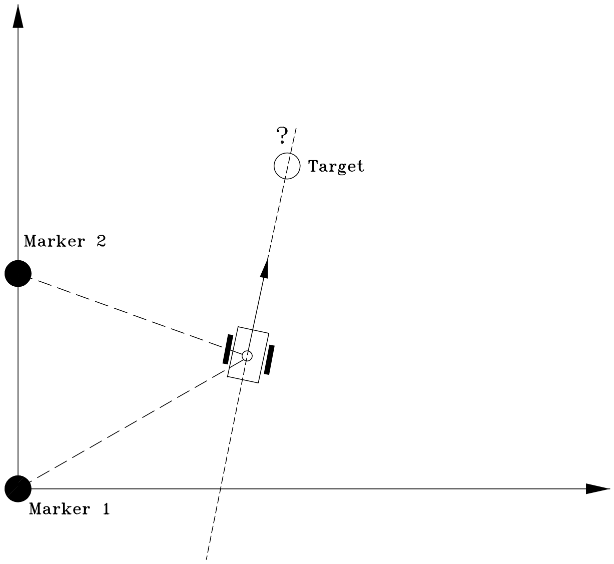

Figure 3: A vehicle that moves along straight lines cannot localize a target if the vehicle aims at the target directly.

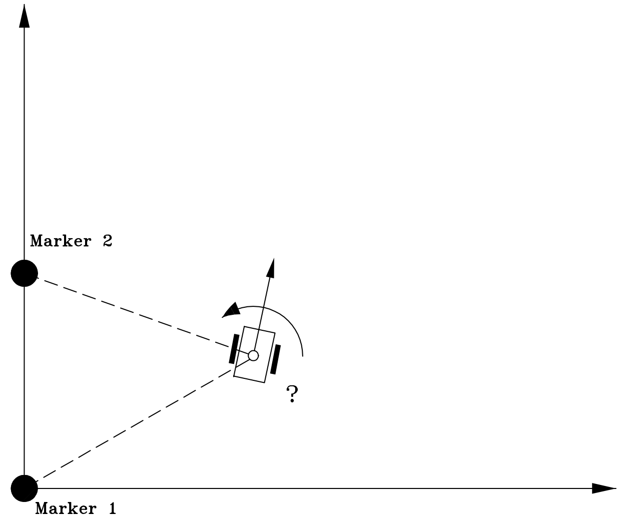

Figure 4: A vehicle that can only rotate cannot localize itself.

gaussian noises. Also let the initial conditions be $x(0) \sim N(\mathbf{x}_o, \mathbf{P}_o)$. For this system, the filter (see e.g. Gelb [3], Misawa and Hedrick [5]) is implemented by constructing an identical model with estimated states $\hat{\mathbf{x}} \in \mathbb{R}^n$,

$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= \mathbf{f}(\hat{\mathbf{x}}, t) + \mathbf{G}(\hat{\mathbf{x}}, t)\mathbf{u}(t) + \mathbf{K}(t)[\mathbf{y} - h(\hat{\mathbf{x}}, t)], \\ \mathbf{K}(t) &= \mathbf{P}(t)\mathbf{H}^T(\hat{\mathbf{x}}, t)\mathbf{R}^{-1}(t) \\ \dot{\mathbf{P}}(t) &= \mathbf{F}(\hat{\mathbf{x}}, t)\mathbf{P}(t) + \mathbf{P}(t)\mathbf{F}^T(\hat{\mathbf{x}}, t) + \mathbf{Q}(t) + \\ & -\mathbf{P}(t)\mathbf{H}^T(\hat{\mathbf{x}}, t)\mathbf{R}^{-1}(t)\mathbf{H}(\hat{\mathbf{x}}, t)\mathbf{P}(t) \end{aligned}$$

where

$$\mathbf{F}(\hat{\mathbf{x}}, t) = \frac{\partial \mathbf{f}(\mathbf{x}, t)}{\partial x},$$

$$\mathbf{H}(\hat{\mathbf{x}}, t) = \frac{\partial \mathbf{h}(\mathbf{x}, t)}{\partial x}.$$

To illustrate problems potentially arising when using this approach to localize a vehicle by triangulation, let us refer to an even further simplified model of the vehicle and of its observations. Consider a holonomic vehicle that can freely translate in the plane where it moves, without process noise (i.e., assuming perfect odometry is available), as

$$\begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{4}$$

This simple vehicle must be localized by triangulation with respect to two markers placed in two distinct points in the plane. Since the vehicle orientation is irrelevant to its motion, only the difference between the angles under which the markers are seen from the vehicle is relevant to localization relative to the markers (i.e., up to a rigid motion of the plane containing the vehicle and the markers). Therefore, we can consider a single measurement output equation as

$$\mathbf{y}' = h'(\mathbf{x}) = atan2((a\mathbf{x}_1), (\mathbf{x}_1^2 + \mathbf{x}_2^2 - a\mathbf{x}_2)) + \mathbf{r}'(t),$$

or simply (excluding the circle of radius $a/2$ centered in $(\mathbf{x}_1 = 0, \mathbf{x}_2 = a/2)$) as

$$\mathbf{y} = h(\mathbf{x}) = \frac{a\mathbf{x}_1}{\mathbf{x}_1^2 + \mathbf{x}_2^2 - a\mathbf{x}_2} + \mathbf{r}(t),$$

where y is the angle between the rays from the vehicle to the two markers. Assume for simplicity $\mathbf{R}(t) \equiv 1$. From fig.5, it appears clearly that a single noiseless measurement would identify the position of the vehicle up to a circle passing trough the two markers. Now consider the construction of an EKF as described above,

$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= \mathbf{u} + \mathbf{P}\mathbf{H}^T(\mathbf{y} - \mathbf{h}(\hat{\mathbf{x}})) \\ \dot{\mathbf{P}} &= -\mathbf{P}\mathbf{H}^T\mathbf{H}\mathbf{P} \end{aligned}$$

where

$$\mathbf{H} = \left[\frac{a(\mathbf{x}_2^2 - \mathbf{x}_1^2) - a^2\mathbf{x}_2}{(\mathbf{x}_1^2 + \mathbf{x}_2^2 - a\mathbf{x}_2)^2} \;\; \frac{(a - 2\mathbf{x}_2)a\mathbf{x}_1}{(\mathbf{x}_1^2 + \mathbf{x}_2^2 - a\mathbf{x}_2)^2} \right].$$

The dynamics of the observation error $\mathbf{e} = \hat{\mathbf{x}} - \mathbf{x}$ are simply $\dot{\mathbf{e}} = \mathbf{P}\mathbf{H}^T[\mathbf{y} - \mathbf{h}(\hat{\mathbf{x}})]$. To verify convergence of the estimates, consider the function $V = \mathbf{e}^T\mathbf{e}$ and its time derivative

$$\dot{V} = \mathbf{e}^T\dot{\mathbf{e}} = (\hat{\mathbf{x}} - \mathbf{x})^T\mathbf{P}\mathbf{H}^T[\mathbf{y} - \mathbf{h}(\hat{\mathbf{x}})] \quad (5)$$
$$= (\hat{\mathbf{x}} - \mathbf{x})^T\mathbf{P}\mathbf{H}^T \left.\frac{\partial h}{\partial x}\right|_{\mathbf{x}=\hat{\mathbf{x}}} (\mathbf{x} - \hat{\mathbf{x}}) + O^3(\|\hat{\mathbf{x}} - \mathbf{x}\|) \quad (6)$$
$$= -\mathbf{e}^T\mathbf{P}\mathbf{H}^T\mathbf{H}\mathbf{e} + O^3(\|\mathbf{e}\|) \quad (7)$$
$$(8)$$

Being $\mathbf{P}$ a symmetric positive definite matrix for all times $t < \infty$, the first term on the right hand side of eq.8 is positive semidifinite, vanishing only when $\mathbf{e}$ lies in the nullspace of $\mathbf{H}$. Therefore, whenever $\mathbf{He} \neq 0$, there exists ϵ such that $\forall \mathbf{e} : \|\mathbf{e}\| < \epsilon$, the non–definite, order–three term on the right hand side is dominated by the quadratic term, and $\|\mathbf{e}\|$ decreases. However, if a trajectory of the system exists such that $\mathbf{e}$ keeps within the nullspace of $\mathbf{H}$, then V, hence $\|\mathbf{e}\|$, may grow indefinitely.

The interpretation of this filtering mechanisms appears clearly from fig.5, when assuming $\mathbf{P} = p\mathbf{I}$ for simplicity. In fact, the direction of $\mathbf{H} = \left.\frac{\partial \mathbf{h}(\mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{x}=\hat{\mathbf{x}}}$ at each estimate $\hat{\mathbf{x}}$ is that of radius of the circle trough the markers and the current estimate, and its sense and magnitude is chosen according to the innovation $\mathbf{y} - \mathbf{h}(\hat{\mathbf{x}})$ and to the gain p. Thus, when $\hat{\mathbf{x}}$ lies on an inner circle than $\mathbf{x}$ (i.e., when $\mathbf{h}(\hat{\mathbf{x}}) > \mathbf{h}(\mathbf{x})$), the estimate is changed so as to move $\hat{\mathbf{x}}$ outwards, and viceversa. Therefore, the filter will not converge if the trajectory followed by the vehicle is a circle through the two markers ($\mathbf{y} \equiv \mathbf{h}(\hat{\mathbf{x}})$).

4 Optimal exploration paths

In the previous section we have seen that in the localization problem, observability and convergence of filters depend upon trajectories followed by the system, and hence ultimately upon controls. Notice that this feature is absolutely peculiar to nonlinear systems, as in the linear case the input has no role to play in the estimation process.

In applications such as the exploration of a planet's soil by a robotic rover, besides the need to locate itself and targets w.r.t. to markers, the system is also confronted with limitations in autonomy of motion, e.g. in the total length of the path the vehicle can track in one day. Therefore, when the localization problem is of paramount importance, the problem arises of not wasting any autonomy in "unuseful" trajectories, and rather

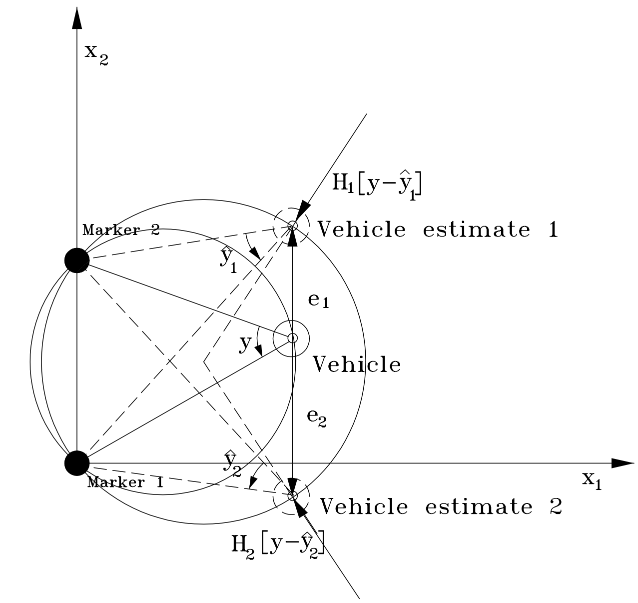

Figure 5: Interpretation of the Extended Kalman filter for localization.

to choose among all the trajectories the one which will maximize the overall estimation accuracy.

We can pose this problem as an optimal control problem as follows: maximize the functional

$$J(\mathbf{u}) = -\left\|\left(\int_0^T \frac{\partial y}{\partial \mathbf{x}_o}^T \frac{\partial y}{\partial \mathbf{x}_o} dt\right)^{-1}\right\| \quad (9)$$

subject to the constraints

$$L = \int_0^T \sqrt{(\dot{\mathbf{x}}_1^2 + \dot{\mathbf{x}}_2^2)}\, dt, \quad (10)$$
$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{u}); \quad \mathbf{x}(0) = \mathbf{x}_o, \quad (11)$$
$$\mathbf{y} = \mathbf{h}(\mathbf{x}). \quad (12)$$

In this formulation, (10) establishes the length of the path to be followed, while (11) and (12) correpond to vehicle dynamics and measurement equations, respectively. The interpretation of the optimality index is given in terms of the following considerations. Consider the output function $\mathbf{y}(t) = \mathbf{h}(\mathbf{x}(t))$ as a function of the initial conditions $\mathbf{x}_o \in \mathbb{R}^2$ and of the input functions $\mathbf{u} \in U$, with U a suitable functional space, and denote this as $\mathbf{y}(\mathbf{x}_o, \mathbf{u}, t)$. Let $\mathbf{x}_o^o$ and $\mathbf{x}_o'$ denote two different initial conditions, with $\|\mathbf{x}_o^o - \mathbf{x}_o'\| < \epsilon$, and write

$$\mathbf{y}(\mathbf{x}_o', \mathbf{u}, t) - \mathbf{y}(\mathbf{x}_o^o, \mathbf{u}, t) = \left.\frac{\partial y}{\partial x_o}\right|_{\mathbf{x}_o=\mathbf{x}_o^o} (\mathbf{x}_o' - \mathbf{x}_o^o) + O^2(\epsilon)$$

In order to distinguish between $\mathbf{x}_o^o$ and $\mathbf{x}_o'$ based on the difference in outputs, premultiply both sides by $\frac{\partial y}{\partial x_o}\Big|_{\mathbf{x}_o=\mathbf{x}_o^o}^T$ (denoted $\frac{\partial \mathbf{y}}{\partial \mathbf{x}_o}$ for short) and integrate from time 0 to T to get

$$\int_0^T \frac{\partial y}{\partial x_o}^T (\mathbf{y}(\mathbf{x}_o', \mathbf{u}, t) - \mathbf{y}(\mathbf{x}_o^o, \mathbf{u}, t))\, dt + O^2(\epsilon)$$

$$= \left(\int_0^T \left(\frac{\partial y}{\partial x_o}^T \frac{\partial y}{\partial x_o} \right) dt \right) (\mathbf{x}_o' - \mathbf{x}_o^o)$$

This equation has the form of a linear system $\mathbf{b}+\delta = \mathbf{A}\mathbf{x}$, where the known vector $\mathbf{b}$ comes from measurement outputs, the perturbation term δ comes from approximations errors (and possibly from measurement noise), and matrix $\mathbf{A}$ depends on inputs. Invertibility of $\mathbf{A}$ is tantamount to observability of the system. Also, in order to have the least propagation of perturbations δ in the solution $\mathbf{x}$, it is well known (see e.g. Bicchi and Canepa [1]) that some norm of the inverse of $\mathbf{A}$ should be minimized.

Notice that the criterion in (9) does not reflect any particular choice in the estimator or filter adopted, and is therefore intrinsic to the reconstructibility of the state from the given trajectory. A characterization of this criterion in terms of the Fisher information matrix and the Cramer–Rao bounds associated to the problem was presented by Piloni and Bicchi [6].

The solution of the above optimal control problem might be difficult in general to obtain. In paticular, notice that the cost functional is not in the standard form $J = \int_0^T L(\mathbf{x}, \mathbf{u}) dt$, and does not enjoy the "localization" property, i.e. additivity over time intervals.

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References

1. Bicchi, A., and Canepa, G.: "Optimal Design of Multivariate sensors", *Measurement Science and Technology*, 5 (1994) 319–332 . IOP Publishing Ltd, 1994.
2. J. Borenstein, A. Everett, and L. Feng: "Navigating mobile robots", AK Peters, 1996.
3. Gelb, A.: "Applied optimal estimation", M.I.T. Press, Cambridge, MA, 1974.
4. Isidori, A.: "Nonlinear Control Systems", 3rd ed., Springer Verlag, 1995.
5. Misawa, E.A., and Hedrick, J.K.: "Nonlinear Observers — A state–of–the–art Survey", *Trans. ASME,* vol. 111, pp. 344–352, 1989.
6. Piloni, F., and Bicchi, A.: "Navigazione di veicoli autonomi mediante visione artificiale: pianificazione dei moti di esplorazione", Internal Report, Centro Piaggio, University of Pisa, 1994.