Hypercubes are minimal controlled invariants for discrete time linear systems with quantized scalar input

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Abstract

Quantized linear systems are a widely studied class of nonlinear dynamics resulting from the control of a linear system through finite inputs. The stabilization problem for these models shall be studied in terms of the so called practical stability notion that essentially consists in confining the trajectories into sufficiently small neighborhoods of the equilibrium (ultimate boundedness).

In this paper, we study the problem of describing the smallest sets into which any feedback can ultimately confine the state, for a given linear single-input system with an assigned finite set of admissible input values (quantization). We show that a controlled invariant set of minimal size is contained in a family of sets (namely, hypercubes in canonical controller form), previously introduced in [14, 15]. A comparison is presented which quantifies the improvement in tightness of the proposed analysis technique with respect to classical results using quadratic Lyapunov functions.

Keywords: controlled invariance, quantized systems, practical stability.

1 Introduction

Quantized control systems are controlled dynamical systems with input and/or output maps taking values in finite sets. As a simple reference model, consider a discrete–time quantized

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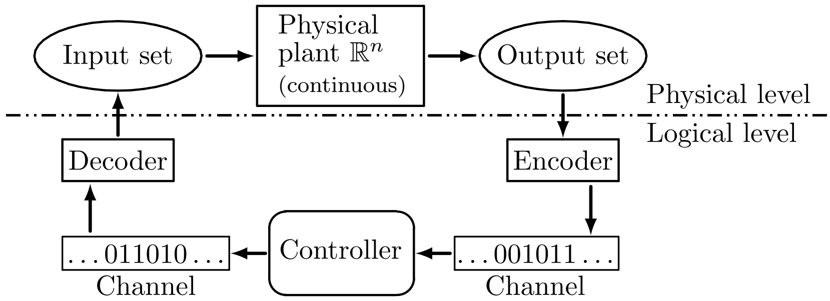

Figure 1: Graphical illustration of the hybrid structure of a quantized control system.

system given by

$$\begin{cases} x(t+1) = f\big(x(t), u(t)\big) \\ y(t) = h\big(x(t)\big) \\ x(t) \in \mathcal{X} = \mathbb{R}^n\,,\ u(t) \in \mathcal{U}\,,\ h(x) \in \mathcal{Y}\,, \end{cases} \tag{1}$$

where $\mathcal{U}$ or $\mathcal{Y}$ are finite sets and $\forall\, u \in \mathcal{U}\,,\ f(\,\cdot\,, u)$ is a smooth function.
The study of these models has increasingly attracted the attention of the control community in the past twenty years $\big($see for instance [6, 21, 5, 7, 2, 3, 17, 20, 10]$\big)$. In fact, the need of dealing with this special class of systems arises from many control applications. For instance, a possible way to study the control of a physical system interfaced by digital sensors and actuators is that of considering mathematical models including a properly chosen quantized output function h and a finite control set $\mathcal{U}$. This approach is particularly suitable when the sensors and/or actuators are "low–cost" (as e.g., a stepper motor) so introducing a coarse quantization. Another commonly encountered example is concerned with the presence of finite capacity communication links in the control loop: in this case quantization must be introduced in order to properly encode analog signals into a finite set of symbols to be transmitted over the communication channel (see Fig. 1).
More in general, in a quantized control problem we have to deal with a hybrid structure which is organized into two levels: at the logical level we perform the control synthesis (the controller is in fact a device manipulating output and input strings from discrete alphabets); at the physical level, the plant is modelled by an equation like (1). The overall picture results in a highly nonlinear dynamical system. Indeed, even in the seemingly easy situation in which the dynamics is described by a linear transformation $\big($i.e., $f(x, u) = Ax + Bu\big)$, the presence of discrete variables produces nonlinear closed loop dynamics which may exhibit features such as the presence of multiple isolated equilibria and chaotic behaviors $\big($see e.g., [6, 8]$\big)$. This kind of models will be referred to as quantized linear systems and will be the subject of this work.

We are interested in the stabilizability problem for discrete–time quantized linear systems.

The basic observation is that if the control function $u(t)$ is constrained to take values from a finite set and the system is open loop unstable, it is not possible to confine the trajectories within arbitrarily small neighborhoods of the origin [6]. Therefore, practical stability notions are to be considered for these models. Accordingly, the practical stabilization may consist in the synthesis of symbolic feedback controllers capable of steering the system to within sufficiently small neighborhoods of the equilibrium, starting from large attraction basins [6, 22, 10, 19]. The size of the final set within which the trajectories are confined is a measure for the steady–state performance of the closed loop dynamics.
Hence, for a given quantized system, the practical stabilizability analysis can be divided in two steps. The first stage is prior to the control synthesis and consists in finding the *controlled invariant* neighborhoods of the equilibrium, namely the sets such that there exists a quantized control law $u(t) \in \mathcal{U}$ ensuring that the trajectories starting from that set remain in the set. Once a proper notion for the *size* of a set is defined, the identification of the *smallest* controlled invariant neighborhood of the equilibrium allows to establish the optimal steady–state performance one could aim at obtaining. In a second stage the control synthesis is performed with the purpose of achieving the convergence of the trajectories towards the smallest invariant set specified in the first stage.
As for the search of controlled invariant sets, it will be satisfactory to identify a sufficiently rich family of sets. In this paper we are mainly interested in the steady–state performance, therefore the suitability of the considered family is related to the existence of a controlled invariant set in the family whose size (in some proper sense) is sufficiently small or even minimal with respect to the size of any other controlled invariant set.

In [14, 15], we have addressed the stabilization problem for n–dimensional discrete–time linear systems under quantization of both the scalar input and output variables. While standard stability analysis techniques are based on the study of quadratic Lyapunov functions and invariant ellipsoids, in that work we found convenient first to change the state space coordinates to the so called canonical controller form, then to perform the stability analysis in terms of invariant hypercubes.
According to the picture drawn above, in this paper we first briefly review the technique based on invariant hypercubes. We hence introduce the notion of size of a set. The main focus is then steered into showing that, for the most interesting types of control quantizers (i.e., the uniform and the logarithmic ones), the family of invariant hypercubes actually contains one element whose size is minimal with respect to all invariant sets. For these quantizers the convergence of the trajectories towards the minimal hypercube can be ensured too. These results guarantee that the hypercubes based analysis is a suitable choice and it is not conservative as far as the steady–state performance is concerned.

In this work, the analysis is restricted to systems with input quantization only (i.e., $y = x$). To reach our purpose, we first analyze some geometric properties holding for invariant sets of arbitrary shape and show the peculiarities exhibited by hypercubes in controller form coordinates. Such a discussion will be helpful to introduce the suitable notions of size of a set. The first notion, called magnitude, essentially consists in measuring the extension of the set in terms of the infinity norm. The corresponding *minimality in magnitude* property is shown to hold for the final invariant hypercube, in the case of uniformly or logarithmically quantized input sets, irrespective of the unstable dynamics of the linear system. Since sets with the same magnitude can have different volumes, the notion of magnitude is then strengthened by adding a requirement which involves the containment relation. This leads to the so called *strong minimality* property that will be shown to hold for the final invariant hypercube provided that the system is sufficiently unstable.
A quantitative comparison is also presented that shows the improvements in the practical stability analysis that can be obtained via the hypercubes based technique with respect to the approach relying on classical Lyapunov theory. Nonetheless, a counterpart of the hypercubes based approach for multi–input systems or continuous–time models is still missing. In these cases the classical approach fully preserves its validity.

There exists a wide literature on problems concerned with controlled invariance (see [4] and references therein). However, most of the results on constrained control are limited to bounded convex sets and hence do not apply to the quantized control case. A quite general framework within which invariant sets can be studied is provided by the Viability theory [1]. Here the limits are concerned with the algorithmic procedures which the main tools are based on (as for instance the "Controlled Invariance Kernel Algorithm" [18]). Typical problems stem from having to cope with the huge computational complexity due to the increasingly larger number of constraints needed to describe the sets involved in the iterative procedures. Another interesting approach is the one proposed in [19] for switching systems (thus including quantized systems as a special case) where invariant Euclidean balls are algorithmically computed using nonlinear programming. This technique can efficiently handle the two stages of practical stabilization (i.e., invariance and convergence); on the other hand it has the drawback of having to cope with a non–convex optimization, hence the global optimum may not be found. Also, because the algorithm is restricted to search for Euclidean balls, it does not provide, in general, a minimal invariant set. In this work instead, thanks to the nice geometric properties of the controller form, it has been possible to perform a precise and non conservative analysis without resorting to algorithmic approaches, hence avoiding the related complexity.

We finally remark that the need to restrict to the notion of practical stability is forced by the choice of considering control laws taking values in a finite set. Other works on quantized systems (see [5, 7, 12, 20]) show the possibility of achieving asymptotic stability by means of time varying control policies (i.e., a finite set of control values is taken to be time varying and adaptively chosen). Nevertheless these techniques result in a control law $u(t)$ taking infinite values accumulating towards the limit point $u = 0$.

The paper is organized as follows: in Section 2 we fix the notation and provide a brief review of the practical stabilization technique introduced in [14, 15], the main results on the optimality of such a technique are presented in Section 3.

Notation and terminology: While $S_1 \subseteq S_2$ refers to the standard containment relation for sets, $S_1 \subset S_2$ means that S_1 is *strictly* contained in S_2, namely $S_1 \subseteq S_2$ and $\exists\, s \in S_2$ such that $s \notin S_1$. A topological set S is said to be *discrete* iff all its points are isolated. Given $E \subseteq \mathbb{R}^k$: ${}^cE$ and $\# E$ denote respectively its complementary and its cardinality; for $v \in \mathbb{R}^k$, let $E - v := \{x \in \mathbb{R}^k \,|\, x + v \in E\}$. Given $E \subseteq \mathbb{R}$, $\mathrm{diam}(E) := \sup\limits_{x,y \in E} |x - y|$ is the diameter of E. Let $x \in \mathbb{R}^n$ and x_i be the i^{th} coordinate of x: given $\Omega \subseteq \mathbb{R}^n$, $\mathrm{Pr}_i\, \Omega := \{\omega_i \in \mathbb{R} \,|\, \omega = (\omega_1, \ldots, \omega_i, \ldots, \omega_n) \in \Omega\}$, $\mathrm{diam}_i\, \Omega := \mathrm{diam}\big(\mathrm{Pr}_i\, \Omega\big)$; if $A \in \mathbb{R}^{n\times n}$, $A\,\Omega = \{A\omega \,|\, \omega \in \Omega\}$. For $x \in \mathbb{R}^n$, $\|x\|$ and $\|x\|_\infty := \max_{i=1,\ldots,n}|x_i|$ stand for the Euclidean and infinity vector norm of x respectively; the corresponding induced matrix norms are denoted by $\|A\|$ and $\|A\|_\infty$. $Q_n(\Delta) := \big[-\frac{\Delta}{2}; \frac{\Delta}{2}\big]^n = \big\{x \in \mathbb{R}^n \,|\, \|x\|_\infty \leq \frac{\Delta}{2}\big\}$ is the closed hypercube of edge length Δ while $Q_n^o(\Delta) := \big[-\frac{\Delta}{2}; \frac{\Delta}{2}\big[^n$ is the semi–open hypercube. x' denotes the transpose of the vector x, $x^+(t)$ stands for $x(t+1)$: the dependance on t will be often omitted. For a given $\mathbb{R}^{n\times n} \ni P > 0$ and $\mathbb{R} \ni r > 0$, let $\mathcal{E}_{P,r} := \{x \in \mathbb{R}^n \,|\, x'Px \leq r\}$.

2 Preliminaries and Review

We are interested in the stabilization issue for the following discrete–time, single–input linear system under quantized control:

$$\begin{cases} x^+(t) = Ax(t) + Bu(t) \\ x \in \mathbb{R}^n, \quad u \in \mathcal{U} \subset \mathbb{R}, \quad t \in \mathbb{N}, \end{cases}$$

where $\mathcal{U}$ is a closed and discrete set containing 0. Such a system will be denoted by the triple $(A, B, \mathcal{U})$. We assume that the pair (A, B) is reachable and, without loss of generality, that the pair (A, B) is in *controller form*:

$$A = \begin{pmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{pmatrix}, \; B = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}, \tag{2}$$

where $s^n - \alpha_n s^{n-1} - \cdots - \alpha_2 s - \alpha_1$ is the characteristic polynomial of A. Because

$$\|A\|_\infty = \max_{i=1,\dots,n} \sum_{j=1}^{n} |A_{i,j}| = \max\Big\{1\,,\, \sum_{i=1}^{n} |\alpha_i|\Big\}\,,$$

if $\sum_{i=1}^n |\alpha_i| \leq 1$ then the system is stable, we hence assume $\sum_{i=1}^n |\alpha_i| > 1$ and we let $\alpha := \sum_{i=1}^n |\alpha_i|$.

As for the control set, because $\mathcal{U}$ is supposed to be closed and discrete, then $\mathcal{U}$ may have countably infinite cardinality. Nevertheless, under this assumption, any bounded set contains only a finite number of elements of $\mathcal{U}$. Hence, control sets having $u = 0$ as an accumulation point are not allowed and, from the point of view of the steady–state analysis, this is not different from assuming $\#\mathcal{U} < +\infty$. For $\mathcal{U} \neq \{0\}$, let

$$u_0 := \min_{u \in \mathcal{U}\setminus\{0\}} |u|\,. \tag{3}$$

The practical stability property we are interested in is based on the notion of invariant set [4]:

Definition 1 *A set $\Omega \subseteq \mathbb{R}^n$ is said to be* positively invariant *for a system $x^+ = f(x)$ iff $\forall\, x \in \Omega\,,\ x^+ \in \Omega\,;$*

Definition 2 *A set $\Omega \subseteq \mathbb{R}^n$ is said to be* controlled invariant *for system $(A, B, \mathcal{U})$ iff $\forall\, x \in \Omega\,,$ $\exists\, u \in \mathcal{U}$ such that $x^+ = Ax + Bu \ \in \Omega\,.$*

Definition 3 (Practical stability) *Let $0 \in \Omega \subseteq X_0 \subseteq \mathbb{R}^n$ with Ω being a bounded neighborhood of 0. A feedback control law $u:\ \mathbb{R}^n \to \mathcal{U}$ is said to be (X_0, Ω)–*stabilizing *iff the corresponding closed loop system $x^+ = Ax + Bu(x)$ is such that both Ω and X_0 are positively invariant and $\forall\, x(0) \in X_0\,,$ $\exists\, t_{x(0)} \in \mathbb{N}$ such that $x(t_{x(0)}) \in \Omega\,.$ If moreover $\forall\, x(0) \in X_0\,,$ $t_{x(0)} \leq m\,,$ then the control law is said to be (X_0, Ω)–*stabilizing in m steps. *A system $(A, B, \mathcal{U})$ is said to be (X_0, Ω)–*stabilizable *iff there exists a (X_0, Ω)–stabilizing control law.*

We shall make use of the quantized version of the so called deadbeat controller:

Definition 4 *Given system $(A, B, \mathcal{U})$, let*

$$\nu\,:\ \mathbb{R} \longrightarrow \mathcal{U}$$

be a mapping which associates to each real number r an element of $\mathcal{U}$ minimizing the Euclidean distance from r. The feedback law $u : \mathbb{R}^n \to \mathcal{U}$ defined by

$$u(x) = \nu\big(-\textstyle\sum_{i=1}^n \alpha_i x_i\big)$$

is called quantized deadbeat controller *(qdb–controller).*

The function ν is well defined because $\mathcal{U}$ is a closed set.

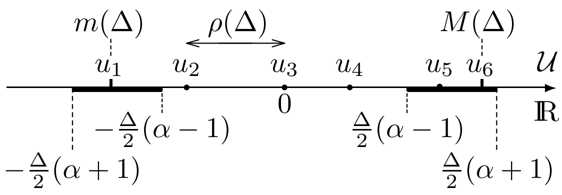

Figure 2: $\mathcal{U}(\Delta) = \{m(\Delta) = u_1, u_2, u_3, u_4, u_5, u_6 = M(\Delta)\}$: $\rho(\Delta) = u_3 - u_2$, the thicker segments represent the intervals where $m(\Delta)$ and $M(\Delta)$ satisfy inequalities (7a–b).

2.1 Review

Hypercubes $Q_n(\Delta)$ in controller form coordinates are particularly suitable to study the practical stability problem. In fact, given $\Delta > 0$, let $x \in Q_n(\Delta)$ and $u \in \mathcal{U}$: by the controller form of (A, B),

$$x^+ = \big(x_2, \ldots, x_n, \textstyle\sum_i \alpha_i x_i + u\big) \in Q_n(\Delta) \;\Leftrightarrow\; \big|\textstyle\sum_i \alpha_i x_i + u\big| \leq \frac{\Delta}{2}\,.$$

Namely, the invariance of $Q_n(\Delta)$ can be tested considering the n^{th} component only.
The following Lemma singles out the control values that are relevant to ensure the invariance of $Q_n(\Delta)$:

Lemma 1 *[15] If $x \in Q_n(\Delta)$ and u is such that $x^+ \in Q_n(\Delta)$, then $u \in \big[-\frac{\Delta}{2}(\alpha+1)\,;\,\frac{\Delta}{2}(\alpha+1)\big]$.* □

Accordingly, we define the finite set

$$\mathcal{U}(\Delta) := \mathcal{U} \cap \big[-\tfrac{\Delta}{2}(\alpha+1)\,;\,\tfrac{\Delta}{2}(\alpha+1)\big]\,. \tag{4}$$

The hypercubes based analysis of practical stability is performed in terms of the following scalar functions of the edge length Δ: let

$$\begin{cases} m(\Delta) := & \min \mathcal{U}(\Delta) \\ M(\Delta) := & \max \mathcal{U}(\Delta) \end{cases} \tag{5}$$

and

$$\rho(\Delta) := \begin{cases} \sup & \big\{b - a \mid \,]a; b[\, \subseteq [\,m(\Delta)\,; M(\Delta)\,] \quad \text{and} \\ & \quad]a; b[\, \cap\, \mathcal{U}(\Delta) = \emptyset\,\big\} \quad \text{if } \#\mathcal{U}(\Delta) > 1 \\ +\infty & \text{(conventionally)} \quad \text{otherwise.} \end{cases} \tag{6}$$

be the dispersion of $\mathcal{U}(\Delta)$ (see Fig. 2). The three functions $m(\Delta)$, $M(\Delta)$, $\rho(\Delta)$ depend on the dynamics of the system only through the infinity norm of A. Given $\mathcal{U} \neq \{0\}$, let $\bar{\Delta} := \frac{2u_0}{\alpha+1}$ (see Equation (3)), then: for $\Delta < \bar{\Delta}$, $\rho(\Delta) = +\infty$; for $\Delta \geq \bar{\Delta}$, $\rho(\Delta)$ is piecewise constant and non–decreasing with Δ.

Theorem 1 (Practical stability) *[15] Consider a quantized linear system* $(A, B, \mathcal{U})$ *.*
ı) For $\Delta > 0$ *,* $Q_n(\Delta)$ *is controlled invariant if and only if*

$$\begin{cases} m(\Delta) \le -\frac{\Delta}{2}(\alpha - 1) & \text{(7a)} \\ M(\Delta) \ge \frac{\Delta}{2}(\alpha - 1) & \text{(7b)} \\ \rho(\Delta) \le \Delta\,. & \text{(7c)} \end{cases}$$

ıı) If $\Delta_0 > 0$ *satisfies the strict version of inequalities* (7) *, let* $\Delta_\star := \max\{\Delta < \Delta_0 \,|\, \rho(\Delta) = \Delta\}$ *, then the qdb–controller is* $\big(Q_n(\Delta_0), Q_n(\Delta_\star)\big)$*–stabilizing.* □

Inequalities (7a-b) ensure that there is enough control authority to achieve the invariance of $Q_n(\Delta)$ and, in the strict version, convergence properties. Inequality (7c) is a bound on the quantization error.

Remark 1 *For a quantized system* $(A, B, \mathcal{U})$ *, if* $Q_n(u_0)$ *is controlled invariant, it is straightforward to verify that for almost all* $x \in Q_n(u_0)$ *, there exists a unique* $u \in \mathcal{U}$ *such that* $x^+ \in Q_n(u_0)$ *and that such a control value coincides with the one selected by the qdb–controller. Namely, the qdb–controller is the only control law that guarantees the positive invariance of* $Q_n(u_0)$ *.*
This is one of the reasons why Theorem 1.ıı , namely the statement on the control synthesis, is concerned with the qdb–controller only. On the other hand, for $x(t) \in \mathbb{R}^n \setminus Q_n(u_0)$ *several controllers can be defined guaranteeing the convergence towards* $Q_n(u_0)$ $\big($*see [13] for more details*$\big)$ *.*

Theorem 1 holds for very general kinds of input sets, indeed the only relevant assumption is that $\mathcal{U}$ is a closed subset of $\mathbb{R}$, also the assumption on the discrete structure of $\mathcal{U}$ can be removed. Uniformly and logarithmically quantized control sets have been often considered in the literature for their interesting features both from the theoretical and the practical point of view. Let us specialize the results of Theorem 1 to these two cases.

Definition 5 *A control set* $\mathcal{U} \subset \mathbb{R}$ *is said to be* uniformly quantized *iff* $\mathcal{U} = u_0\mathbb{Z}$ *for some* $u_0 > 0$ *.*

Example 1 (Uniform quantization) *Given a system* $(A, B, \mathcal{U})$ *, if* $\mathcal{U}$ *is uniformly quantized it is not difficult to show that* $\forall\, \Delta \ge \bar{\Delta} = \frac{2u_0}{\alpha+1}$ *,* $\rho(\Delta) = u_0$ *and that* $\forall\, \Delta \ge u_0$ *, the hypercube* $Q_n(\Delta)$ *is controlled invariant and the qdb–controller is* $\big(Q_n(\Delta), Q_n(u_0)\big)$*–stabilizing in* n *steps.* ♣

Definition 6 *A control set* $\mathcal{U} \subset \mathbb{R}$ *is said to be* logarithmically quantized *iff* $\mathcal{U} = \{0\} \cup \{\pm\,\theta^n u_0 \,|\, n \in \mathbb{N}\}$ *for some* $\theta > 1$ *and* $u_0 > 0$ *.*

Example 2 (Logarithmic quantization) *Given a system* $(A, B, \mathcal{U})$, *let us suppose that the control set is logarithmically quantized. According to Equations (4), (5) and (6), we have:*

$$\rho(\Delta) = \begin{cases} +\infty & \text{if} \quad \Delta < \frac{2u_0}{\alpha+1} \\ u_0 \quad \text{if} \quad \frac{2u_0}{\alpha+1} \le \Delta < \frac{2u_0}{\alpha+1}\, \theta^{\left\lceil \log_\theta\left(\frac{\theta}{\theta-1}\right)\right\rceil} \\ u_0 \frac{\theta-1}{\theta} \cdot \theta^{\left\lfloor \log_\theta \left(\frac{(\alpha+1)\,\Delta}{2u_0}\right)\right\rfloor} & \text{otherwise,} \end{cases}$$

and

$$M(\Delta) = -m(\Delta) = \begin{cases} 0 \quad \text{if} \quad \Delta < \frac{2u_0}{\alpha+1} \\ u_0 \cdot \theta^{\left\lfloor \log_\theta \left(\frac{(\alpha+1)\,\Delta}{2u_0}\right)\right\rfloor} & \text{otherwise.} \end{cases}$$

For $\Delta \ge \frac{2u_0}{\alpha+1}\, \theta^{\left\lceil \log_\theta\left(\frac{\theta}{\theta-1}\right)\right\rceil}$,

$$\rho(\Delta) \le \frac{(\theta-1)(\alpha+1)}{2\,\theta}\,\Delta\,,$$

and for $\Delta \ge \frac{2u_0}{\alpha+1}$,

$$M(\Delta) > u_0 \cdot \theta^{\log_\theta \left(\frac{(\alpha+1)\,\Delta}{2u_0}\right) - 1} = \frac{\alpha+1}{2\,\theta}\,\Delta$$

(both the inequalities are tight). The case in which $1 < \theta < \frac{\alpha+1}{\alpha-1}$ *is particularly interesting. In fact, using the expressions above, it is easy to check that* $\forall\,\Delta > u_0$, $\rho(\Delta) < \Delta$; $\rho(u_0) = u_0$ *and* $\forall\,\Delta \ge u_0$, $M(\Delta) > \frac{\Delta}{2}(\alpha-1)$. *Therefore, Theorem 1 ensures that* $\forall\,\Delta \ge u_0$, $Q_n(\Delta)$ *is controlled invariant and the qdb–controller is* $\big(Q_n(\Delta), Q_n(u_0)\big)$*–stabilizing.* ♣

3 Geometric properties of invariant sets and minimality properties of hypercubes

There are many possible shapes for invariant sets, the reasons for considering one class or another (e.g., ellipsoids, hypercubes or more general polytopes) can be varied. Three basic requirements one would aim at satisfying are: simplicity of description of the considered sets, simplicity of practical stability analysis and optimality. As discussed in the introduction, since the goal of practical stabilization is to confine the trajectories of the system within *small* controlled invariant neighborhoods of the equilibrium, then optimality means that the considered family contains an invariant set within which trajectories can be made to converge and that the size of such a set is *minimal* with respect to all controlled invariant sets. These requests are often trading off: e.g., ellipsoids can be easily described but they are not optimal (as it will be shown in Section 3.3); polytopes instead are usually optimal but may be of arbitrarily complex description.

In this section we first point out the simplicity of the practical stability analysis based on hypercubes, hence we analyze geometric properties holding for invariant sets of arbitrary shape and show the peculiarities exhibited by hypercubes in controller form coordinates. This analysis is helpful to properly define the concept of size for a set and is introductory to the statement of the minimality theorems for hypercubes. The bottom line is that the choice of considering hypercubes for the practical stabilization problem is motivated by the fact that they meet all the requirements of simplicity of description, simplicity of analysis and optimality.
The simplicity of description of hypercubes is apparent, as well as the resulting practical stability analysis (see Theorem 1). The following simple result is helpful to appreciate such a simplicity compared with other types of sets:

Lemma 2 *[9] $\Omega \subseteq \mathbb{R}^n$ is controlled invariant if and only if $A\Omega \subseteq \bigcup_{u\in\mathcal{U}} (\Omega - Bu)$.* □

Despite the simple formulation, the practical application of this invariance criterion is not straightforward when dealing with arbitrary sets Ω. In particular, to test the invariance of Ω, it is in general necessary to determine $A\Omega$. We have seen instead that for $\Omega = Q_n(\Delta)$ the analysis can be reduced to a 1–dimensional problem where invariant hypercubes are characterized by simple algebraic relations between Δ, α and the scalar functions $\rho(\Delta)$, $m(\Delta)$ and $M(\Delta)$. Furthermore, while Lemma 2 may give some insight on the geometric characteristics of controlled invariant sets, on the other hand it does not really answer the question of how to construct controlled invariant sets for a given system $(A, B, \mathcal{U})$.

3.1 Geometric properties of invariant sets

The attention is now turned to the study of some geometric properties holding for arbitrarily shaped invariant sets and on how these results can be used for the practical stability analysis. Since we will widely exploit the properties of the canonical controller form, it is worth recalling that the control acts only on the n^{th} component while the others shift upwards.
Let $\Pr_{(i_1,\ldots,i_m)} x := (x_{i_1}, \ldots, x_{i_m})$:

Proposition 1 *If $\Omega \subseteq \mathbb{R}^n$ is controlled invariant, then $\Pr_{(2,\ldots,n)} \Omega \subseteq \Pr_{(1,\ldots,n-1)} \Omega$. In particular, $\Pr_n \Omega \subseteq \Pr_{n-1} \Omega \subseteq \cdots \subseteq \Pr_1 \Omega$ and $\operatorname{diam}_n \Omega \leq \operatorname{diam}_{n-1} \Omega \leq \ldots \leq \operatorname{diam}_1 \Omega$.*

Proof. $\forall\, y = (y_2, \ldots, y_n) \in \Pr_{(2,\ldots,n)} \Omega$, $\exists\, x \in \Omega$ with $x = (x_1, y_2, \ldots, y_n)$. Let $u \in \mathcal{U}$ be such that $x^+ \in \Omega$: $x^+ = (y_2, \ldots, y_n, x_n^+)$, hence $y \in \Pr_{(1,\ldots,n-1)} \Omega$. ■

Given $\Omega \subseteq \mathbb{R}^n$, let $\mathcal{Z} := \Pr_n \Omega$ and $\Omega^* := \Omega \cap \mathcal{Z}^n$. The main property of Ω^* is exhibited by the following

Proposition 2 *If* $\Omega \subseteq \mathbb{R}^n$ *is a controlled invariant neighborhood of the origin, then* Ω^* *is a controlled invariant neighborhood of the origin and* $\forall\,\phi:\ \mathbb{R}^n \to \mathcal{U}$ *rendering* Ω *positively invariant,* ϕ *is* (Ω, Ω^*)*–stabilizing in* $n-1$ *steps.*

Proof. Ω^* is a neighborhood of the origin because such are both Ω and $\mathcal{Z}^n$. Since Ω is positively invariant, to prove that ϕ is (Ω, Ω^*)–stabilizing in $n-1$ steps, we have to show that $\forall\, x(0) \in \Omega$, $\forall\, t \geq n-1$ and $\forall\, i = 1, \ldots, n$, $x_i(t) \in \mathcal{Z}$. Indeed, $\forall\, x \in \Omega$, $x_n^+ \in \mathcal{Z}$ by definition of $\mathcal{Z}$. Since the system is in controller form, the thesis follows. ■

Corollary 1 *If* ϕ *is* (X_0, Ω)*–stabilizing, then* ϕ *is* (X_0, Ω^*)*–stabilizing.* □

Therefore, $\Omega \setminus \Omega^*$ is a redundant part of the invariant set Ω, meaning that the trajectories lie within $\Omega \setminus \Omega^*$ only for a transient time of at most $n-1$ steps. As the aim of practical stabilization is to confine the trajectories within small controlled invariant neighborhoods of the origin, in the analysis of stabilizing control laws it is then proper to replace the final set Ω by Ω^*, namely to "*cut off*" the redundant region. This procedure is effective because, by Proposition 1, $\mathrm{Pr}_i\,\Omega \subseteq \mathrm{Pr}_{i-1}\,\Omega \quad \forall\, i = 2, \ldots, n$ and $\Omega \setminus \Omega^* \neq \emptyset$ whenever one of the inclusions is strict. In general, $(\Omega^*)^* \subset \Omega^*$, namely the cut–off procedure can be iterated.

Notice that if $\Omega = Q_n(\Delta)$, then $\Omega^* = \Omega$, namely the hypercubes $Q_n(\Delta)$ are non–redundant. This is not the case for more commonly encountered types of invariant sets such as ellipsoids. Quantitative results on the effect of the cut–off procedure on ellipsoids will be given in Section 3.3.

Hypercubes are not the only example of invariant sets such that $\Omega^* = \Omega$, the same property holds if Ω is inscribed in a hypercube. This fact will be relevant in next section when discussing on the minimality of hypercubes and is related with the advisability of introducing two notions of minimality. Indeed, among these notions, the strongest one will allow to exclude the existence of invariant sets Ω inscribed in the smallest invariant hypercube.

3.2 Minimality properties of invariant hypercubes

In order to investigate the minimality properties of the smallest controlled invariant hypercube with respect to all controlled invariant sets, we need to introduce a suitable notion of size for controlled invariant sets. We choose to study the minimality problem by comparing sets according to their extension in some vector norm $\|\cdot\|_*$. That is, for a neighborhood Ω of the origin, we consider $\|\Omega\|_* := \sup\limits_{x \in \Omega} \|x\|_*$. Indeed, by achieving the convergence of the trajectories to within Ω, it is guaranteed that $\limsup_{t \to +\infty} \|x(t)\|_* \leq \|\Omega\|_*$.

For comparison purposes, we will also consider the volume and the containment relation. Nevertheless, the volume only is not suitable in the practical stability framework because it does not provide any information about how far a trajectory can go away from the equilibrium. As for the containment relation, although it may appear to be a natural way of comparing invariant sets, this relation is not a total ordering and controlled invariance does not behave well as for intersection (i.e., if Ω_1 and Ω_2 are controlled invariant, then not necessarily $\Omega_1 \cap \Omega_2$ is controlled invariant), therefore the minimality problem formulated in terms of the containment relation is not well posed.

In what follows, sets are measured by considering their extension in the infinity norm in the controller form coordinates. More precisely, we consider the diameter of the sets along the n coordinate directions (i.e., $\operatorname{diam}_i \Omega$, $i = 1, \ldots, n$). Actually, according to Propositions 1 and 2, the relevant quantity is $\operatorname{diam}_n \Omega$, in fact longer extensions of Ω along the other directions can be cut off. We hence give the following

Definition 7 *Consider a system $x^+ = Ax + Bu$ in the controller form coordinates and let Ω be a controlled invariant neighborhood of the origin: the quantity $\operatorname{diam}_n \Omega$ will be referred to as the* magnitude *of Ω. We will say that Ω is* minimal in magnitude *iff any bounded controlled invariant neighborhood of the origin Ω' has a magnitude greater than or equal to that of Ω.*

If the pair (A, B) is not in controller form, the magnitude of Ω can be easily calculated through the formula $\operatorname{diam}\big(\Pr_B(\Omega)\big)/\|B\|^2$, where $\Pr_B(x) := x'B \in \mathbb{R}$.

Hence, the magnitude is the measure we will use for comparing invariant sets. In some cases it will be still possible to consider the containment relation and to study minimality properties which are stronger than minimality in magnitude: a result in this vein will be given in next Theorem 3.

Obviously, if A is a stable matrix, there exist arbitrarily small invariant neighborhoods of the origin: therefore we will be interested only in the case of unstable matrices.

Theorem 2 (Minimality in magnitude) *Let $u_0 = \min\limits_{u \in \mathcal{U} \setminus \{0\}} |u|$ and Ω be a bounded controlled invariant neighborhood of the origin. If A is an unstable matrix, then $\operatorname{diam}_i \Omega \geq u_0 \ \forall\, i = 1, \ldots, n$. In particular, if $Q_n(u_0)$ is controlled invariant, then $Q_n(u_0)$ is minimal in magnitude.*

Proof. Thanks to Proposition 1, it is sufficient to show that $\operatorname{diam}_n \Omega \geq u_0$ (i.e., that the magnitude of Ω is greater than or equal to u_0). Let us assume by contradiction that $d := \operatorname{diam}_n \Omega < u_0$. Set $a_1 := \inf\limits_{x \in \Omega} x_n$ and $a_2 := \sup\limits_{x \in \Omega} x_n$, then $\Pr_n \Omega \subseteq [\, a_1 \,; a_2 \,]$, $a_2 - a_1 = d < u_0$ and $0 \in [\, a_1 \,; a_2 \,]$. Let Ω_0 be the path connected component of Ω containing 0. As $\Pr_n \circ A$ is a continuous function, $\Pr_n(A\Omega_0)$ is an interval. Two cases can occur:

I) Suppose that $\Pr_n(A\Omega_0) \cap {}^c[\, a_1 \,; a_2 \,] \neq \emptyset$: since $\Pr_n(A\Omega_0)$ is an interval that intersects $[\, a_1 \,; a_2 \,]$ (in fact it contains 0), then, with $\theta := u_0 - d$, there exists $\hat{x} \in \Omega_0$ such that $(A\hat{x})_n \in \,]\, a_1 - \theta \,; a_1 \,[\,\cup\,]\, a_2 \,; a_2 + \theta \,[$. In this case, by the definition of u_0, it is easy to see that $\forall\, u \in \mathcal{U}$,

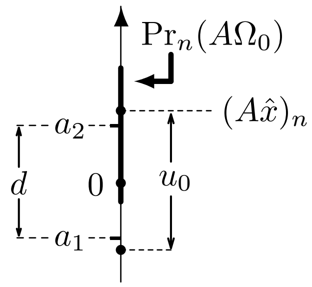

Figure 3: Visual help for the proof of Theorem 2 : the thicker segment represents $\mathrm{Pr}_n\,(A\Omega_0)$.

$\hat{x}_n^+ \notin [\,a_1\,;a_2\,]$ (see Fig. 3) : this contradicts the controlled invariance of Ω as $\mathrm{Pr}_n\,\Omega \subseteq [\,a_1\,;a_2\,]$.
II) Suppose instead that $\mathrm{Pr}_n\,(A\Omega_0) \subseteq [\,a_1\,;a_2\,]$. We claim that $\exists\, x \in \Omega_0$ such that $Ax \notin \Omega$. The claim implies the thesis, in fact: for such an x , by the controlled invariance of Ω , $\exists\, u \in \mathcal{U} \setminus \{0\}$ such that $x^+ \in \Omega$, but $u \neq 0$ together with $(Ax)_n \in [a_1\,;\,a_2]$ and $a_2 - a_1 < u_0$ imply $x_n^+ \notin [\,a_1\,;a_2\,]$ which contradicts the fact that $x^+ \in \Omega$.
Let us prove the claim: first, since Ω_0 is a bounded neighborhood of the origin, $A\Omega_0 \not\subseteq \Omega_0$. In fact, if the contrary held, then $\forall\, k \in \mathbb{N}$, $A^k\Omega_0 \subseteq \Omega_0$ which contradicts the fact that A is unstable. Since $A\Omega_0$ is path connected, if $A\Omega_0 \subseteq \Omega$, then $A\Omega_0$ would be contained in a path connected component of Ω . As $0 \in A\Omega_0 \cap \Omega_0$, then $A\Omega_0 \subseteq \Omega_0$ which is a contradiction. ■

Corollary 2 *If system $(A, B, \mathcal{U})$ is reachable and A is unstable, a necessary condition for the (X_0, Ω)–stabilizability of the system is that the magnitude of Ω is greater than or equal to u_0 .* □

Clearly, for the (X_0, Ω)–stabilizability it is also necessary that Ω is reachable from X_0 . The general reachability issue is not faced here. However, the cases in which $\mathcal{U}$ is uniformly or logarithmically quantized provide examples where the (X_0, Ω)–stabilizability holds with $\Omega = Q_n(u_0)$. Namely, the lower bound for the magnitude is attained by a hypercube. In fact, for uniformly quantized controls $\mathcal{U} = u_0\,\mathbb{Z}$, the minimal invariant hypercube is $Q_n(u_0)$ which, by Theorem 2 , is minimal in magnitude. Furthermore, $\forall\, \Delta \geq u_0$, the system is $\big(Q_n(\Delta),\, Q_n(u_0)\big)$–stabilizable (see Example 1) . In the logarithmically quantized case, for $\mathcal{U} = \{0\} \cup \{\pm\, \theta^n u_0 \,|\, n \in \mathbb{N}\}$, with $1 < \theta \leq \frac{\alpha+1}{\alpha-1}$ and $u_0 > 0$, the same property holds for $Q_n(u_0)$ (see Example 2) .
It is worth noting that invariant neighborhoods Ω strictly contained in $Q_n(u_0)$ and with smaller volume can exist (see Example 3 below) . Nevertheless, Theorem 2 states that, even if such an Ω exists, it spreads up to the border of $Q_n(u_0)$ in all the directions of the coordinate axes (i.e., $\forall\, i = 1, \ldots, n$, $\mathrm{diam}_i\,\Omega = u_0$; see Fig. 4) so that Ω and $Q_n(u_0)$ are equivalent as for their extension in the infinity norm. Therefore, even if the convergence of the trajectories to within such an Ω was proved, no improvement would be obtained in terms of the asymptotic behavior

of the system, meaning that it would be still guaranteed that $\limsup_{t\to+\infty}\|x(t)\|_\infty \le \|\Omega\|_\infty = \|Q_n(u_0)\|_\infty$.

Example 3 *Let us consider the quantized control system*

$$\begin{cases} x^+ = Ax + Bu \\ x \in \mathbb{R}^n,\ u \in \mathbb{Z}\,, \end{cases}$$

where, as usual, the pair (A,B) is in controller form. It is easy to see that the semi–open hypercube $Q_n^o(1) = \left[-\frac{1}{2};\frac{1}{2}\right[^n$ is controlled invariant and that $\forall\, x \in Q_n^o(1)$, there exists a unique $u \in \mathbb{Z}$ such that $x^+ \in Q_n^o(1)$. It is hence univocally defined the map

$$\begin{array}{rccc} T: & Q_n^o(1) & \to & Q_n^o(1) \\ & x & \mapsto & x^+ \end{array},$$

where $x^+ = Ax + B\,u(x)$ and $u(x) \in \mathbb{Z}$.
Assume that A is an unstable matrix such that $0 < |\det A| < 1$.
Since A is unstable, $Q_n^o(1)$ is minimal in magnitude by Theorem 2. Because $\det A \neq 0$, T is a local diffeomorphism at 0, therefore, $\forall\, k \in \mathbb{N}$, the set $T^k\big(Q_n^o(1)\big)$ is a neighborhood of the origin. Moreover, since $T^{k+1}\big(Q_n^o(1)\big) \subseteq T^k\big(Q_n^o(1)\big)$, then $T^k\big(Q_n^o(1)\big)$ is controlled invariant and, being a subset of $Q_n^o(1)$, it is minimal in magnitude. Furthermore, we claim that

$$\forall\, k \in \mathbb{N}, \quad T^{k+1}\big(Q_n^o(1)\big) \subset T^k\big(Q_n^o(1)\big) \tag{8}$$

and, denoted by λ the Lebesgue measure,

$$\lim_{k\to+\infty} \lambda\Big(T^k\big(Q_n^o(1)\big)\Big) = 0\,. \tag{9}$$

Namely, $\{T^k\big(Q_n^o(1)\big)\}_{k\in\mathbb{N}}$ is a strictly decreasing sequence of controlled invariant neighborhoods of the origin made of minimal in magnitude sets and containing elements of arbitrarily small volume. According to Theorem 2, all of these sets spread up to the border of $Q_n(1)$ in all the coordinate directions, thus having the same extension in the infinity norm as $Q_n(1)$. The typical structure of one of the sets of the sequence (in the two dimensional case) is represented by the shaded region in Fig. 4.
Before proving the claim, notice that the set $T^k\big(Q_n^o(1)\big)$ is reachable in k steps by any point in $Q_n^o(1)$ and, with a qdb–controller, in $k+n$ steps by any point in $\mathbb{R}^n$ (see Example 1).
Let us prove the claim. As for the inclusion (8), because $T^{k+1}\big(Q_n^o(1)\big) \subseteq T^k\big(Q_n^o(1)\big)$, we have only to show that indeed the inclusion is strict. It holds that

$$\forall\, k \in \mathbb{N}, \quad \lambda\Big(T^{k+1}\big(Q_n^o(1)\big)\Big) \le \lambda\Big(\big(A \circ T^k\big)\big(Q_n^o(1)\big)\Big)\,. \tag{10}$$

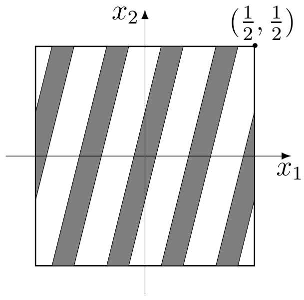

Figure 4: The non–connected shaded region represents the controlled invariant set $T\big(Q_n^o(1)\big)$ for the two dimensional system discussed in Example 3 and having $\alpha_1 = 0.4$ and $\alpha_2 = 4$.

In fact: $\forall\, u \in \mathbb{Z}$, *let* $\mathcal{S}_u := \big\{x \in \mathbb{R}^n \,|\, u-\frac{1}{2} \le x_n < u+\frac{1}{2}\big\}$ *and* $\mathcal{R}_u := \mathcal{S}_u \cap \big(A\circ T^k\big)\big(Q_n^o(1)\big)$ *; then* $\big(A\circ T^k\big)\big(Q_n^o(1)\big) = \bigcup_{u\in\mathbb{Z}} \mathcal{R}_u$ *and* $T^{k+1}\big(Q_n^o(1)\big) = \bigcup_{u\in\mathbb{Z}} (\mathcal{R}_u - Bu)$, *inequality* (10) *then easily follows. Since* $\lambda\Big(\big(A\circ T^k\big)\big(Q_n^o(1)\big)\Big) = |\det A| \cdot \lambda\Big(T^k\big(Q_n^o(1)\big)\Big)$ *and* $|\det A| < 1$, *inequality* (10) *yields*

$$\forall\, k \in \mathbb{N}, \quad \lambda\Big(T^{k+1}\big(Q_n^o(1)\big)\Big) \le |\det A| \cdot \lambda\Big(T^k\big(Q_n^o(1)\big)\Big) < \lambda\Big(T^k\big(Q_n^o(1)\big)\Big).$$

This implies that $\forall\, k \in \mathbb{N}$, $T^{k+1}\big(Q_n^o(1)\big) \subset T^k\big(Q_n^o(1)\big)$ *and* $\lambda\Big(T^k\big(Q_n^o(1)\big)\Big) \le |\det A|^k \cdot \lambda\big(Q_n^o(1)\big)$, *thus the limit in Equation* (9) *holds*. ♣

Example 3 shows that a minimal in magnitude set can contain other minimal in magnitude sets having smaller volume (indeed, having an arbitrarily small volume). This raises the need for introducing the concept of strong minimality which strengthens the minimality in magnitude by involving the containment relation.

Definition 8 *A controlled invariant neighborhood of the origin* Ω *is said to be* strongly minimal *iff it is minimal in magnitude and any neighborhood of the origin* Ω' *strictly contained in* Ω *is not controlled invariant.*

If the system is sufficiently unstable (in a sense specified below), then the strong minimality property holds for hypercubes. More precisely,

Theorem 3 (Strong minimality) *Let* $u_0 = \min\limits_{u\in\mathcal{U}\setminus\{0\}} |u|$. *If* $|\alpha_1| > 1 + \sum_{i=2}^n |\alpha_i|$ *and* $\Omega \subseteq Q_n^o(u_0)$ *is a controlled invariant neighborhood of the origin, then* $\Omega = Q_n^o(u_0)$. *In particular, if* $Q_n^o(u_0)$ *is controlled invariant, then it is strongly minimal.*

Proof. We show that if such an Ω exists, then it contains a subset whose uncontrolled evolution is confined within $Q_n^o(u_0)$ until it covers the whole semi–open hypercube. By definition of u_0,

such an evolution is also the unique ensuring that the trajectories starting from this subset remain within $Q_n^o(u_0)$: since Ω is controlled invariant, this entails that $\Omega = Q_n^o(u_0)$.
In detail, the matrix A is invertible and

$$(A^{-1}x)_j = \begin{cases} \frac{x_n - \sum_{i=1}^{n-1} \alpha_{i+1} x_i}{\alpha_1} & \text{if } j = 1 \\ x_{j-1} & \text{otherwise}. \end{cases} \tag{11}$$

Let $\theta := \frac{(1+\sum_{i=2}^n |\alpha_i|)}{|\alpha_1|}$. By the hypothesis $\theta < 1$, then

$$\forall\, x \in \mathbb{R}^n, \quad \left|(A^{-1}x)_1\right| \leq \frac{|x_n| + \sum_{i=1}^{n-1} |\alpha_{i+1}||x_i|}{|\alpha_1|} \leq \theta \cdot \|x\|_\infty < \|x\|_\infty . \tag{12}$$

Equations (11) and (12) imply that $A^{-1}Q_n^o(u_0) \subset Q_n^o(u_0)$, thus $\forall\, h \in \mathbb{N}$

$$A^{-h}Q_n^o(u_0) \subseteq A^{-h+1}Q_n^o(u_0) \subseteq \cdots \subseteq A^{-1}Q_n^o(u_0) \subset Q_n^o(u_0), \tag{13}$$

and in particular $\|A^{-h}\|_\infty \leq 1$. Moreover, by the Hamilton–Cayley identity, $A^{-n} = \frac{1}{\alpha_1}\big(I - \sum_{i=2}^n \alpha_i A^{-n-1+i}\big)$, therefore $\|A^{-n}\|_\infty \leq \theta$: this means that $A^{-n}Q_n(u_0) \subseteq Q_n(\theta\, u_0)$ and it immediately follows that $\forall\, k \in \mathbb{N}$, $A^{-nk}Q_n^o(u_0) \subseteq Q_n(\theta^k\, u_0)$.
Let Ω be a controlled invariant neighborhood of the origin: since $\lim_{k\to+\infty} \theta^k = 0$, $\exists\, \hat{k} \in \mathbb{N}$ such that $Q_n(\theta^{\hat{k}}\, u_0) \subseteq \Omega$, therefore $A^{-n\hat{k}}Q_n^o(u_0) \subseteq \Omega$. We claim that if $A^{-m}Q_n^o(u_0) \subseteq \Omega$ for some $m \geq 1$, then $A^{-m+1}Q_n^o(u_0) \subseteq \Omega$. In our case the hypothesis of the claim is satisfied $\forall\, m \geq n\hat{k}$ and the recursive application of the claim implies that $Q_n^o(u_0) \subseteq \Omega$, namely the thesis.
Let us prove the claim. First, we show that if $x \in \Omega$ and $Ax \in Q_n^o(u_0)$, then $Ax \in \Omega$. In fact, by the controlled invariance of Ω, $\exists\, u \in \mathcal{U}$ such that $x^+ \in \Omega \subseteq Q_n^o(u_0)$: such a control value must be $u = 0$ because for $u \neq 0$, $x^+ \notin Q_n^o(u_0)$. Indeed, $-\frac{u_0}{2} \leq (Ax)_n < \frac{u_0}{2}$ by assumption, hence

$$-\frac{u_0}{2} + u \leq x_n^+ = (Ax)_n + u < \frac{u_0}{2} + u, \tag{14}$$

and for $u \neq 0$ it holds that $|u| \geq u_0$ which, together with inequalities (14), yields either $x_n^+ \geq \frac{u_0}{2}$ or $x_n^+ < -\frac{u_0}{2}$. Now, consider $y \in A^{-m+1}Q_n^o(u_0)$ and let us show that $y \in \Omega$: since $-m+1 \leq 0$, then $y \in Q_n^o(u_0)$ (see the inclusions in Equation (13)). Let $x := A^{-1}y \in A^{-m}Q_n^o(u_0)$: $x \in \Omega$ by assumption and $y = Ax \in Q_n^o(u_0)$, therefore $y \in \Omega$. $\blacksquare$
By Theorem 3, $Q_n^o(u_0)$ is strongly minimal in both the cases of uniformly and logarithmically quantized controls (provided that in the latter case we assume $1 < \theta \leq \frac{\alpha+1}{\alpha-1}$).

Remark 2 *Assuming* $|\alpha_1| > 1 + \sum_{i=2}^n |\alpha_i|$, *which by the way is a condition involving only the coefficients of the characteristic polynomial of* A, *is the same as asking that* $A^{-1}Q_n^o(u_0) \subset Q_n^o(u_0)$, *namely it is a stability requirement on the matrix* A^{-1}, *hence corresponding to an instability property of* A.

It can be shown that the condition ensuring the strong minimality of $Q_n^o(u_0)$ is only sufficient, nevertheless the result is interesting because it shows that there are cases in which, among the minimal diameter sets (i.e., $\mathrm{diam}_i\,\Omega = u_0 \;\; \forall\, i = 1,\ldots,n$), the whole $Q_n^o(u_0)$ is actually the smallest controlled invariant set.

3.3 Hypercubes vs ellipsoids

In this section, single–input systems under uniformly quantized controls are considered and a quantitative comparison between different practical stability analysis tools is provided. It is shown in particular that the classical quadratic Lyapunov functions based approach yields significantly more conservative results than those obtained by considering the effect of the cut–off procedure and, above all, by the hypercubes based analysis. The case of two dimensional systems is considered in full details. Some results are presented for the general case too and suggest that, as the state space dimension increases, the classical Lyapunov approach is more and more conservative.
Let us consider the quantized control system

$$\begin{cases} x^+ = Ax + Bu \\ x \in \mathbb{R}^2,\; u \in \mathbb{Z}\,, \end{cases} \tag{15}$$

where the pair (A,B) is in controller form $\big($see Equation (2)$\big)$. Let $K := (-\alpha_1 \;\; -\alpha_2)$ and $u(x) = \nu(Kx)$ be the qdb–controller. With $e(x) := \nu(Kx) - Kx$, the closed–loop dynamics is

$$\begin{cases} x^+ = (A+BK)x + Be(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e(x) := Fx + Be(x) \\ |e(x)| \le \frac{1}{2}\,. \end{cases} \tag{16}$$

Practical stability for system (16) follows by the fact that an open loop asymptotically stable linear system is *input–to–state* stable [11]. The analysis of practical stability based on hypercubes (squares in this case) says that $\forall\,\Delta \ge 1$, system (16) is $\big(Q_2(\Delta), Q_2(1)\big)$–stable (see Example 1). The following result provides a quantitative analysis based on classical Lyapunov arguments:

Lemma 3 *[16], cf. [5] Consider the system*

$$x^+(t) = Fx(t) + Be(t)\,, \tag{17}$$

where F is Schur (i.e., all its eigenvalues have magnitude strictly less than 1) and $\forall\, t \ge 0$, $\|e(t)\| \le E_0$. For any $\mathbb{R}^{n\times n} \ni S > 0$, let P be the solution of the Lyapunov equation $F'PF -$

$P = -S$ and

$$\begin{cases} r_{\mathrm{i}} := R^2\big(\lambda_{\max}(P-S) + \lambda_{\min}(S)\big)\,, \ \textit{where} \\ R = \frac{E_0}{\lambda_{\min}(S)}\,\alpha(P)\,, \ \textit{and} \\ \alpha(P) = \ \|F'PB\| + \sqrt{\|F'PB\|^2 + \lambda_{\min}(S)\|B'PB\|}\,. \end{cases}$$

Then,

$\imath$) $\forall\, r \geq r_{\mathrm{i}}$, $\mathcal{E}_{P,r}$ is invariant;

$\imath\imath$) $\forall\, r_1 \geq r_2 > r_{\mathrm{i}}$ system (17) is $\big(\mathcal{E}_{P,r_1}, \mathcal{E}_{P,r_2}\big)$–stable. □

If system (17) is single–input and in controller form, then, according to Corollary 1, Lemma 3.$\imath\imath$ can be refined by stating that

$\imath\imath$ bis) $\forall\, r_1 \geq r_2 > r_{\mathrm{i}}$ system (17) is $\big(\mathcal{E}_{P,r_1}, \mathcal{E}^*_{P,r_2}\big)$–stable.

Let us apply Lemma 3 to the practical stability analysis of system (16). Let

$$\mathbb{R}^{2\times 2} \ni S = \begin{pmatrix} s_1 & s_3 \\ s_3 & s_2 \end{pmatrix} > 0\,,$$

with $s_1 > 0$, $s_2 > 0$ and $s_1 s_2 - s_3^2 > 0$. We can assume without loss of generality that $s_3 \geq 0$. It holds that: $\lambda_{\min}(S) = \frac{s_1+s_2-\sqrt{(s_1-s_2)^2+4s_3^2}}{2}$,

$$P = \begin{pmatrix} s_1 & s_3 \\ s_3 & s_1+s_2 \end{pmatrix}\,,$$

$R = \frac{1}{2\,\lambda_{\min}(S)}\big(s_3 + \sqrt{\lambda_{\min}(S)^2 + s_1 s_2}\,\big)$ and

$$r_{\mathrm{i}}(S) = R^2\big(s_1 + \lambda_{\min}(S)\big)\,.$$

For a given S, the minimal invariant ellipse provided by Lemma 3 is $\mathcal{E}_{P(S),r_{\mathrm{i}}(S)}$: a possible optimality criterion to select S is given by the minimization of the area of this ellipse, namely

$$\min_{\mathbb{R}^{2\times 2}\ni S>0} \frac{\pi\, r_{\mathrm{i}}(S)}{\sqrt{\det P(S)}}\,.$$

It can be proved that the minimal value of the area is $\frac{\pi}{\sqrt{2}}$ which is achieved for $S = s_1 I$.

For $S = I$, one obtains $r_{\mathrm{i}} = 1$ and

$$\mathcal{E}_{P(I),r_{\mathrm{i}}(I)} = \{x \in \mathbb{R}^2 \,|\, x_1^2 + 2x_2^2 \leq 1\}\,.$$

Let us quantify the effect of the cut–off procedure in terms of area reduction (see also Fig. 5): it holds that $\mathrm{Pr}_2\big(\mathcal{E}_{P(I),r_{\mathrm{i}}(I)}\big) = \Big[-\frac{\sqrt{2}}{2}\,;\,\frac{\sqrt{2}}{2}\Big]$ whereas $\mathrm{Pr}_1\big(\mathcal{E}_{P(I),r_{\mathrm{i}}(I)}\big) = [-1\,;\,1]$, therefore $\mathcal{E}^*_{P(I),r_{\mathrm{i}}(I)} = \mathcal{E}_{P(I),r_{\mathrm{i}}(I)} \cap Q_2(\sqrt{2})$ and[1]

$$\mathrm{Area}\,\big(\mathcal{E}^*_{P(I),r_{\mathrm{i}}(I)}\big) = \frac{4}{\sqrt{2}}\int_0^{\sqrt{2}/2} \sqrt{1-x^2}\,dx = \frac{\pi/2+1}{\sqrt{2}}\,.$$

[1] Indeed, $\int \sqrt{1-x^2}\,dx = \frac{1}{2}\big(x\sqrt{1-x^2} + \arcsin x\big)$.

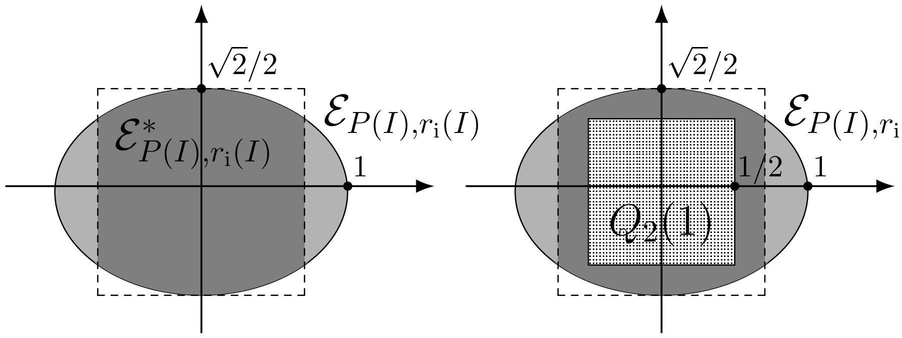

Figure 5: *Left :* The minimal invariant ellipse provided by Lemma 3 with the optimal choice of $S = I$ and, in darker grey, the smaller invariant set $\mathcal{E}^*_{P(I),r_{\mathrm{i}}(I)}$ obtained via the cut–off procedure. *Right:* The dotted region represents the minimal invariant hypercube (square).

Hence,

$$\frac{\mathrm{Area}\,(\mathcal{E}^*_{P(I),r_{\mathrm{i}}(I)})}{\mathrm{Area}\,(\mathcal{E}_{P(I),r_{\mathrm{i}}(I)})} = \frac{1}{2} + \frac{1}{\pi} \simeq 0.82\,.$$

As for the hypercubes based analysis, the area reduction is really significant, in fact

$$\frac{\mathrm{Area}\,\big(Q_2(1)\big)}{\mathrm{Area}\,(\mathcal{E}^*_{P(I),r_{\mathrm{i}}(I)})} = \frac{\sqrt{2}}{\pi/2+1} \simeq 0.55\,.$$

Furthermore, it can be proved that $\forall\, S > 0\,,\ Q_2(1) \subset \mathcal{E}_{P(S),r_{\mathrm{i}}(S)}$. This is in agreement with the minimality properties holding for hypercubes and presented in the previous section.
We conclude with a brief analysis for the generalization to n–dimensional state space of the system in Equation (15) controlled by the qdb–controller. Also in this case we know that $\forall\,\Delta \geq 1$, the closed loop system is $\big(Q_n(\Delta), Q_n(1)\big)$–stable (see Example 1). For $S = I \in \mathbb{R}^{n\times n}$, the minimal invariant ellipsoid provided by the Lyapunov practical stability analysis based on Lemma 3 is

$$\mathcal{E}_{P(I),r_{\mathrm{i}}(I)} = \left\{x \in \mathbb{R}^n \mid \textstyle\sum_{j=1}^n j x_j^2 \leq \frac{n^2}{4}\right\},$$

and, for $j = 1, \ldots, n$, $\mathrm{Pr}_j\big(\mathcal{E}_{P(I),r_{\mathrm{i}}(I)}\big) = \big[-\frac{n}{2\sqrt{j}}\,;\,\frac{n}{2\sqrt{j}}\big]$. In particular, $\mathrm{diam}_n\,\mathcal{E}_{P(I),r_{\mathrm{i}}(I)} = \sqrt{n}$. Therefore, as the state space dimension n increases, while the magnitude of the minimal invariant hypercube remains constant equal to 1, the magnitude of $\mathcal{E}_{P(I),r_{\mathrm{i}}(I)}$ diverges.
A qualitative result for the analysis of the effect of the cut–off procedure can be obtained by considering the ratio between $\mathrm{diam}_1\,\mathcal{E}_{P(I),r_{\mathrm{i}}(I)}$ and $\mathrm{diam}_n\,\mathcal{E}_{P(I),r_{\mathrm{i}}(I)}$. Indeed, $\mathcal{E}^*_{P(I),r_{\mathrm{i}}(I)} = \mathcal{E}_{P(I),r_{\mathrm{i}}(I)} \cap \Big(\mathrm{Pr}_n\big(\mathcal{E}_{P(I),r_{\mathrm{i}}(I)}\big)\Big)^n$, so that $\mathrm{diam}_n\,\mathcal{E}_{P(I),r_{\mathrm{i}}(I)}$ is the magnitude of the invariant set $\mathcal{E}^*_{P(I),r_{\mathrm{i}}(I)}$ and dictates the entity of the "cut", while, according to Proposition 1, the direction along the first

coordinate is the one more affected by the cut–off procedure. Because

$$\frac{\operatorname{diam}_1 \mathcal{E}_{P(I),r_{\rm i}(I)}}{\operatorname{diam}_n \mathcal{E}_{P(I),r_{\rm i}(I)}} = \sqrt{n}\,,$$

also the effect of the cut–off procedure is more and more significant at the increasing of the state space dimension.
The same phenomenon is pointed out when analyzing the volume of the minimal invariant ellipsoid. In fact, such a volume, which is to be compared with the unitary volume of the minimal invariant hypercube $Q_n(1)$, is[2]

$$\text{Volume}\left(\mathcal{E}_{P(I),r_{\rm i}(I)}\right) = \frac{(n\sqrt{\pi})^n}{2^n\,\Gamma\left(\frac{n}{2}+1\right)\sqrt{n!}} := \mathrm{V}(n)$$

and, using the Stirling's formula to bound $n!$, it can be seen that $\lim_{n\to+\infty} \mathrm{V}(n) = +\infty$.

Remark 3 *In the above analysis uniformly quantized controls have been considered. Other types of quantizations can be analyzed: in fact, a result analogous to that of Lemma 3 and holding for a wide class of quantizers, including logarithmic ones, has been obtained in [16] by taking advantage of small–gain theory. In this case the study still relies on Lyapunov arguments but a Riccati equation is involved rather than the standard Lyapunov one.*

4 Conclusion

In this work we have studied the problem of describing the smallest neighborhood of the equilibrium into which the trajectories of a linear single–input system with assigned quantized controls can be confined. We have introduced a suitable notion of size of a set and, for a given input quantization, we have provided a lower bound for the minimal feasible size of an invariant set. Such a bound is shown to be attained by a hypercube in controller form coordinates in the master cases of uniform and logarithmic input quantizations. This means that the choice of considering hypercubes is optimal as far as the analysis of the steady–state performance in the practical stabilization problem is concerned.
The multi–input case is more complex because of the lack of a simple canonical form, however we already have some preliminary results based on similar ideas.
In [14, 15], hypercubes have been profitably used also for the analysis of practical stability in the presence of input–and–output quantization. In this more general case, the size of the minimal invariant hypercube is increased by a positive quantity directly related to the output quantizer

[2]Recall that the volume of the unit n–ball is $\frac{\pi^{n/2}}{\Gamma(n/2+1)}$, where $\Gamma(x) := \int_0^{+\infty} e^{-t} t^{x-1} dt$ and $\Gamma(n+1) = n!$.

resolution. However, the problem of characterizing the minimal invariants when both the inputs and the outputs are quantized is open.

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