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A local-local planning algorithm for rolling objects*

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Abstract

In this paper, we consider planning motions of objects of regular shape rolling on a plane among obstacles. Theoretical foundations and applications of this type of operations in robotic manipulation and locomotion have been discussed elsewhere. In this paper, we propose a novel algorithm that improves upon existing techniques in that: i) it is finitely computable and predictable (an upper bound on the computations necessary to reach a given goal within a tolerance can be given), and ii) it possesses a topological (local-local) property which enables obstacles and workspace limitations to be dealt with in an effective way.

1 Introduction

Relocation and reorientation of objects by rolling has been studied extensively in robotics in recent years, for its implications in robotic manipulation ([1, 2, 3, 4, 5, 6]) and robotic locomotion ([7, 8, 9, 10]). In this paper we describe a technique for planning rolling motions of general objects with smooth surface on a plane. We consider a kinematic model of rolling, and assume that the angular velocity components of the rolling body on the plane can be arbitrarily chosen. Previous methods proposed for planning motions of such systems include techniques for particular cases (typically, for a rolling sphere, see e.g. [4]), and very general iterative methods such as the generic loops method of Sontag [11], or the continuation method of Sussmann and Chitour [12]. A technique was proposed in [13] which effectively reduced the problem of planning for general surfaces to the solution of a system of two nonlinear algebraic equations in two unknowns.

All the above methods share two intrinsic limitations. Firstly they are, in one guise or another, iterative methods whose convergence rate is typically slow and hard to predict (no general exact planning method is known at the state of the art). Secondly, they do not consider the possible presence of obstacles. Planning among obstacles is a crucial problem in robotic applications, because of either the presence of physical obstacles in the workspace of a rolling mobile robot, or of constraints imposed by joint limits and physical boundaries of the fingers in manipulating devices.

The problem of planning for nonholonomic systems among obstacles was attacked by [14] with an iterative method derived from those of [11, 12]. A general approach to the problem was considered by [15], who introduced a general topological property of exact planning algorithms in free space, capable of guaranteeing their applicability to planning problems in constrained spaces.

Our aim in this paper is to devise a planning algorithm for rolling manipulation that i) can guarantee convergence to within a given tolerance of the desired final configurations in a finite and predictable number of steps, and ii) can be applied in the presence of constraints in the configuration space.

The basic ingredient of the planner we propose is a lattice structure we superimpose to the configuration space of the rolling system. The lattice, whose mesh size can be adjusted to suit the required accuracy, is obtained by choosing a finite number of basic actions (which could be regarded as control "atoms" or "quanta") to be taken on the system, and considering the effects on the system of applying all the (countably infinite) possible different sequences of such actions. The problem of steering on this lattice will then be solved by constructing a suitable sequence of control quanta. This technique was inspired by similarity with the solution obtained to the planning problem for rolling polyhedral parts, where the quantized nature of control inputs is intrinsic (see [16, 17]).

The rest of this paper is organized as follows: in section 2 we slightly generalize the definition of the topological property of [15] to approximate planning algorithms, and discuss its applications to nonholonomic systems in constrained configuration spaces. In section 3 we describe the general structure of our proposed algorithm; section 4 introduces the basic geometric construction underpinning the algorithm, and section 5 contains the proof of the fact that the proposed algorithm indeed has the invoked topological property.

2 Planning nonholonomic systems among obstacles

Consider the problem of steering a nonholonomic system on a manifold $\mathcal{M}$ between two configurations $p_0, p_{goal} \in \mathcal{M}$, through a trajectory which is admissible with respect to both restrictions on the workspace and nonholonomic constraints. A possible approach is to find first a solution to the (much simpler) problem obtained by removing the nonholonomic constraints, and then to find an approximation to that solution that satisfies the nonholonomic constraint, while keeping away from obstacles.

Assume that the initial and final configurations belong to the same connected component of the free configuration space $\mathcal{C}_{free}$, which is assumed to be an open set. Assume further that a trajectory (or *geometric path*) $\gamma : [0,1] \mapsto \mathcal{M}$, $\gamma(0) = p_0$, $\gamma(1) = p_{goal}$ results from a global planner, such that $\gamma(t) \in \mathcal{C}_{free}$, $t \in [0,1]$. The approximating nonholonomic path Γ is in general

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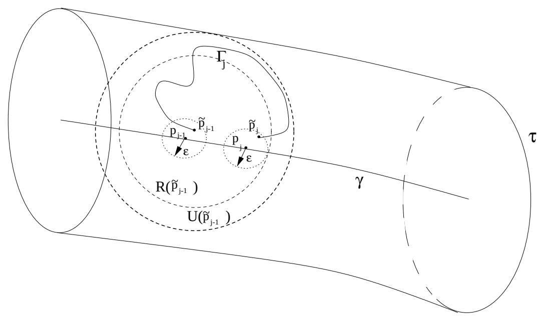

Figure 1: Iterative application of a local–local planner provides an approximation of a geometric path that verifies both nonholonomic and configuration–space constraints.

comprised of a finite concatenation of subpaths

$$\Gamma_i : [0,1] \mapsto \mathcal{M},\ i = 1, \ldots, N,$$

where $\Gamma_i(1) = \Gamma_{i+1}(0)$, $\Gamma_1(0) = p_0$ and $\Gamma_i(1) \in V(p_i)$ with $V(p_i)$ a neighborhood of a point p_i on γ. We do not insist here that the approximating local planner is exact, because such property is not enjoyed by any known planner for rolling motion. However, for all i we assume that the local planner output Γ_i is

- feasible with respect to the nonholonomic constraints, and
- *local–local*, i.e. if the initial and final points of Γ_i are close enough, then Γ_i does not exit a small neighborhood of the initial point.

Denoting by $B(p, \epsilon)$ a ball centered in p of radius ϵ, a more precise definition of the latter property is as follows:

Definition 1 *A local planning algorithm is* local–local *if for any initial configuration p_0 and for all neighborhoods $U(p_0)$ of p_0, there exists a* locally reachable neighborhood *$R(p_0) \subset U(p_0)$ such that, for any goal configuration $p_1 \in R(p_0)$ and for all $\varepsilon > 0$, the algorithm provides a trajectory $\hat{\Gamma} : [0,1] \mapsto \mathcal{M}$ steering the system from $p_0 = \hat{\Gamma}(0)$ to $\tilde{p}_1 = \hat{\Gamma}(1) \in B(p_1, \varepsilon)$ with $\hat{\Gamma}(t) \in U(p_0)\ \forall t \in [0,1]$.*

Such local–local property is clearly a relaxed version of the "topological property" introduced by [15] for exact local planners.

Let a tubular neighborhood τ of the geometric path γ be defined as

$$\tau = \cup_{p \in \gamma} U(p)$$

with $U(p) = B(p, \epsilon(p))$, with $\epsilon(p)$ bounded away from zero (i.e., $\epsilon(p) \geq \epsilon > 0, \forall p \in \gamma$), and assume that $\tau \subset \mathcal{C}_{free}$. Under the assumptions above, a nonholonomic path Γ in the free configuration space can be computed by iteratively applying a local–local algorithm as follows (see fig.2):

1. Denote $U(p_0) \subset \tau$ a neighborhood of p_0 entirely contained in the free configuration space, and let $R(p_0)$ be the corresponding local reachability neighborhood of definition 1;

2. Choose $p_1 = \gamma(t)$ with $t = \max\{s : B(\gamma(s), \varepsilon) \subset R(p_0)\}$ and compute $\Gamma_1 : p_0 \mapsto \tilde{p}_1$ with $\tilde{p}_1 \in B(p_1, \varepsilon)$ and $\Gamma_1(t) \in U(p_0)\ \forall t \in [0,1]$;

3. For all $j \geq 2$, denote $U(\tilde{p}_{j-1}) \subset \tau$ a neighborhood of $\tilde{p}_{j-1}$ and $R(\tilde{p}_{j-1})$ the associated local reachability neighborhood. Choose $p_j = \gamma(t)$ with $t = \max\{s : B(\gamma(s), \varepsilon) \subset R(\tilde{p}_{j-1})\}$ and compute $\Gamma_j : \tilde{p}_{j-1} \mapsto \tilde{p}_j$ with $\tilde{p}_j \in B(p_j, \varepsilon)$ and $\Gamma_j(t) \in U(\tilde{p}_{j-1})\ \forall t \in [0,1]$.

Observe that the procedure above terminates if γ is such that there exists $\delta > 0$ such that $\inf\{\text{diam}\ (R(p)),\ p \in B(\gamma, \delta)\} = \bar{\varepsilon} > 0$. If this is the case the variable ε of the algorithm must be smaller than $\bar{\varepsilon}$.

3 A local-local planner for rolling manipulation

We consider planning motions of an object of general shape on a planar surface. The kinematic equations of rolling motion [1, 18, 13] describe the evolution of the (local) coordinates of the contact point on the surface, u and on the plane, x, along with the (holonomy) angle ψ between two Gauss frames at the contact point, as they change according to the rigid relative motion of the finger and the object described by the relative velocity v and angular velocity $\omega = (\omega_x, \omega_y, \omega_z)$:

$$\begin{array}{rcl} \dot{x} & = & \bar{w} \\ \dot{u} & = & M_o^{-1} R_\psi \bar{w} \\ \dot{\psi} & = & T_o R_\psi \bar{w}, \end{array} \tag{1}$$

with $\bar{w} \stackrel{def}{=} R_\psi K_o^{-1} R_\psi \begin{bmatrix} -\omega_y \\ \omega_x \end{bmatrix}$, $R_\psi = \begin{bmatrix} \cos\psi & -\sin\psi \\ -\sin\psi & -\cos\psi \end{bmatrix}$ and M_o, K_o, T_o denoting the metric, curvature, and torsion forms of the rolling surface.

Let $d(\cdot,\cdot)$ denote the distance between points $p = (u, x, \psi)$ in the state manifold for our problem $\mathcal{M} = \mathbb{R}^2 \times \mathbb{R}^2 \times S^1$. Set $p_0 = (u_0, x_0, \psi_0) \in \mathcal{M}$ and $p_{goal} = (u_{goal}, x_{goal}, \psi_{goal}) \in B(p_0, \bar{\delta}_\mathcal{M})$, with $\bar{\delta}_\mathcal{M} \in \mathbb{R}$ a (small) positive number.

We propose to steer approximately the system between p_0 and p_{goal} through the following intermediate steps:

$$p_0 \stackrel{Step1}{\mapsto} p_a = (u_{goal}, x_a, \psi_a)$$

$$p_a \stackrel{Step2}{\mapsto} p_b = (u_{goal}, x_b, \tilde{\psi}_{goal})$$

$$p_b \stackrel{Step3}{\mapsto} \tilde{p}_{goal} = (u_{goal}, \tilde{x}_{goal}, \tilde{\psi}_{goal})$$

with $\tilde{p}_{goal} \in B(p_{goal}, \varepsilon)$.

These steps will be described and analyzed in detail in the following. To substantiate our claim that the algorithm can steer a rolling body to an arbitrarily small neighborhood of the desired final configuration using a finite number of maneuvers and satisfying the local–local property, we need to introduce some further notation.

Denote

$$\delta_u, \delta_x, \delta_\psi \in \mathbb{R}, \quad \text{positive with } \delta_u^2 + \delta_x^2 + \delta_\psi^2 \leq \delta_{\mathcal{M}}^2, \quad (2)$$

$$\bar{\delta}_u, \bar{\delta}_x, \bar{\delta}_\psi \in \mathbb{R}, \quad \text{positive with } \bar{\delta}_u^2 + \bar{\delta}_x^2 + \bar{\delta}_\psi^2 \leq \bar{\delta}_{\mathcal{M}}^2 \quad (3)$$

and, finally,

$$\varepsilon_u^2, \varepsilon_x^2, \varepsilon_\psi^2 \in \mathbb{R}, \quad \text{positive with } \varepsilon_u^2 + \varepsilon_x^2 + \varepsilon_\psi^2 \leq \varepsilon_{\mathcal{M}}^2; \quad (4)$$

The local–local property of the algorithm will be proved by showing that the resulting trajectory $(u(t), x(t), \psi(t))$ is such that if $u_{goal} \in B(u_0, \bar{\delta}_u)$, $x_{goal} \in B(x_0, \bar{\delta}_x)$ and $\psi_{goal} \in B(\psi_0, \bar{\delta}_\psi)$, then $\forall t$, $u(t) \in B(u_0, \delta_u)$, $x(t) \in B(x_0, \delta_x)$ and $\psi(t) \in B(\psi_0, \delta_\psi)$ with

$$(\delta_u, \delta_x, \delta_\psi) \mapsto 0 \;\text{ for }\; (\bar{\delta}_u, \bar{\delta}_x, \bar{\delta}_\psi) \mapsto 0. \quad (5)$$

Step 1 The first step consists simply in applying to system (1) a constant control $\bar{w}(t) = \bar{w}_a$, $0 \leq t \leq t_a$ such that $u_0 \mapsto u(t_a) = u_{goal}$ exactly. The corresponding trajectory on the plane is a straight line and its length is equal to the Riemannian distance on the surface between u_0 and u_{goal}. Then if $d(u_0, u_{goal}) \leq \bar{\delta}_u$ we set $\delta_u = \delta_x = \bar{\delta}_u$. By geometric analysis of the system equations (see e.g. [13], the infinitesimal variation ψ is equal to the infinitesimal variation of the angle between the coordinate direction of the object surface and the tangent to the curve. Then in a small neighborhood of u_0 the total variation of $\psi(t)$ is bounded and this bound decreases to zero with $\bar{\delta}_u$.

Step 2 Consider the set $\mathcal{L}$ of closed, simple paths of length ϱ on the object surface. Let θ denote the direction of the curve at the initial point u_{goal} in the reference frame of the plane of the finger. Recall that rolling an object on a plane along a closed path on the object surface produces a change in the final orientation of the object which is given by the holonomy angle, i.e. the integral of the gaussian curvature comprised in the portion of region bounded by the curve. In the following section (4) a particularly useful subset $\mathcal{R} \subset \mathcal{L}$ will be introduced, from which an element corresponding to a pair $(\bar{\theta}, \bar{\varrho})$ can be chosen such that the corresponding holonomy angle $\Delta\psi \leq \frac{\varepsilon_\psi}{2}$. Then there exists an integer N (the integer part of $\frac{|\psi_a - \psi_{goal}|}{\Delta\psi}$) such that $\|\psi_a + N\Delta\psi - \psi_{goal}\| \leq \varepsilon_\psi$ and the corresponding closed path $(\bar{\theta}, \bar{\varrho})$ applied N times steers the system in a configuration $p_b = (u_{goal}, x_b, \tilde{\psi}_{goal})$ with $\tilde{\psi}_{goal} = \psi_a + N\Delta\psi \in B(\psi_{goal}, \varepsilon_\psi)$. The proof of the local–local property of this trajectory is postponed to section 5.

Step 3 By this step, the system is steered to some configuration $(u_{goal}, \tilde{x}_{goal}, \tilde{\psi}_{goal})$ with $\|\tilde{x}_{goal} - x_{goal}\| \leq \varepsilon_x$ by applying the following method: observe that $\mathcal{L}$ is a subgroup of the fundamental group of closed paths with base point u_{goal}, i.e. $\mathcal{L}$ is closed under concatenation and inverse. Consider the map Ξ:

$$\Xi : \mathcal{L} \longrightarrow \mathbb{R}^2 \times S^1$$

where $\Xi(l) = (v, \Delta\psi)$, where v denotes the total translation of the contact point on the plane, and $\Delta\psi$ is the net change in the orientation of the object that are obtained corresponding to applying to the rolling object a motion that makes the contact point on the object follow a closed path (i.e., an element of $\mathcal{L}$). For this map Ξ the following properties hold:

a) let l_1 and l_2 be any two paths in $\mathcal{L}$ with $\Xi(l_1) = (v_1, \Delta\psi_1)$ and $\Xi(l_2) = (v_2, \Delta\psi_2)$, then for their concatenation we have (in exponential notation)

$$\Xi(l_2 \circ l_1) = (v_2 e^{i\Delta\psi_1} + v_1, \Delta\psi_2 + \Delta\psi_1).$$

The first component of $\Xi(l_2 \circ l_1)$ is the sum of two vectors of $\mathbb{R}^2$ taking into account that the orientation of the reference frame on the tangent plane at contact point u_{goal} has changed by $\Delta\psi_1$ after that the execution of l_1 is completed. The second component denote the total change of orientation produced by the two paths.

b) As the length l of the path goes to zero, $\Xi(l) \mapsto (0, 0)$. Indeed, Ξ describes the end point map of a smooth differential system with piecewise continuous input, for which continuity of solutions is given by classical results.

Now, consider the existence of closed paths $l \in \mathcal{L}$ that achieve translations of the object on the plane, without changing its orientations, or in other terms, such that

$$\Xi(l) = (v, 0). \quad (6)$$

By the composition law given above, it is clear that such paths exist: indeed, for instance, any element of the commutator subgroup of $\mathcal{L}$ defined as

$$[\mathcal{L}] = \{[l_1, l_2] = l_1^{-1} \circ l_2^{-1} \circ l_1 \circ l_2,\ l_1, l_2 \in \mathcal{L}\}$$

satisfies equation 6. Let $\hat{\mathcal{L}} \subset \mathcal{L}$ denote the set of such paths, and consider a finite subset $\{\hat{l}_i, i = 1, M\}$, each corresponding to a pure translation of the object in the plane by a quantity $\hat{v}_i \in \mathbb{R}^2$. By concatenating (in arbitrary order) such paths taken an integer number of times, i.e. $\alpha_1 \hat{l}_1 \circ \alpha_2 \hat{l}_2 \circ \cdots \alpha_M \hat{l}_M$, $\alpha_i \in \mathbb{Z}$, a net translation of the object is obtained that is given by

$$\hat{v}(\alpha) = \alpha_1 \hat{v}_1 \circ \alpha_2 \hat{v}_2 \circ \cdots \alpha_M \hat{v}_M \quad (7)$$

In other words, the object can be translated by any integer combination of the 2-vectors $\hat{v}_i$, that play the role of control quanta in our planning scheme. It is well known from the theory of linear integer programming, that the set of achievable translations resulting from (7) is a lattice of points in the plane[1].

Such lattice can be made arbitrarily fine. Indeed, assume (it will be proved in section 4) that there exist paths $l_i^j \in \mathcal{L}$, $i = a, b$, $j = 1, 2$ such that

$$\|\Xi(l_i^j)_1 = v_i^j\| \leq \varepsilon_x \;\; i = a, b, j = 1, 2 \quad (8)$$

Then, paths $\hat{l}_i = [l_i^1, l_i^2] \in \hat{\mathcal{L}}$, $i = a, b$ are such that $\max\{\frac{\|\hat{v}_a - \hat{v}_b\|}{2}, \frac{\|\hat{v}_a + \hat{v}_b\|}{2}\} \leq \varepsilon_x$.

[1] we assume that all $\hat{v}_i$ have rational components

In conclusion, a suitable linear integer combination of elementary paths $\hat{l}_i$ can be easily found (by standard ILP techniques such as Hermite normal forms, see e.g. [19]) that steers the system to some configuration $(u_{goal}, \tilde{x}_{goal}, \tilde{\psi}_{goal})$ with $\|\tilde{x}_{goal} - x_{goal}\| \leq \varepsilon_x$. A bound on the number of elementary paths that guarantee convergence to within the desired tolerance can be provided (see [20] where the same techniques are used in planning for polyhedral objects). The local–local property of this kind of trajectories will be shown in Section 5.

4 Geodesic Quadrilaterals

A particular class of closed paths can be used such that the geometrical properties of the system are exploited and the resulting trajectories can be easily computed. Closed paths on the object's surface that are comprised of segments of geodesic curves have the following very useful properties:

- the corresponding trajectory on the finger is piecewise linear. Indeed, if the contact point traces a geodesic curve on one surface of a rolling pair, then it also traces a geodesic on the second surface (see e.g. [21]). On the flat finger surface, geodesics are straight lines;
- each linear segment of the trajectory on the finger has the same length as the corresponding geodesic segment on the object surface; the angle between adjacent segments is the same on the two surfaces;
- by the Gauss-Bonnet theorem, the total change in the holonomy angle ψ due to a piecewise geodesic path correspond to the defect to 2π of the sum of the angles between adjacent geodesic segments.

Consider then a *geodesic quadrilateral* $R(\theta, \varrho)$ comprised of four segments of geodesics on the object's surface. The first point chosen for the construction is the base point u_{goal} (i.e., the desired point of contact on the object surface reached after the first step of the proposed algorithm).

Recall that the object surface is parametrized by $f : \mathbb{R}^2 \mapsto \mathbb{R}^3$. A geodesic quadrilateral $R(\theta, \varrho)$ of *size* $\varrho \in \mathbb{R}$ and *orientation* $\theta \in S^1$ is built as follows (see fig. 2):

1. let $\varphi_1(s)$ be a geodesic curve such that

$$\varphi_1(0) = u_{goal} \quad \text{and} \quad \frac{\dot{\varphi}_1(0)}{\|\dot{\varphi}_1(0)\|}^T \frac{\hat{f}_u}{\|f_u\|} = \cos\theta;$$

2. let $\varphi_1^\perp(s)$ a geodesic curve such that

$$\varphi_1^\perp(0) = u_{goal} \quad \text{and} \quad \frac{\dot{\varphi}_1^\perp(0)}{\|\dot{\varphi}_1^\perp(0)\|}^T \frac{\dot{\varphi}_1(0)}{\|\dot{\varphi}_1(0)\|} = 0;$$

3. let $\varphi_1(t_1) = u_1$ and $\varphi_1^\perp(t_1') = u_1'$ be two points on the surface such that their Riemannian distance from u_{goal} is ϱ;

4. let $\varphi_2(s)$ a geodesic curve such that

$$\varphi_2(0) = u_1' \quad \text{and} \quad \frac{\dot{\varphi}_1^\perp(t_1')}{\|\dot{\varphi}_1^\perp(t_1')\|}^T \frac{\dot{\varphi}_2(0)}{\|\dot{\varphi}_2(0)\|} = 0;$$

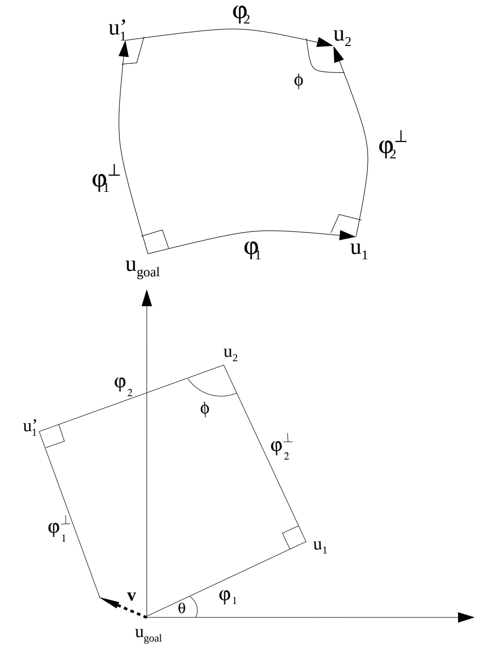

Figure 2: The geometric construction of a geodesic rectangle (top) and its trace on the plane (bottom)

5. let $\varphi_2^\perp(s)$ a geodesic curve such that

$$\varphi_2^\perp(0) = u_1 \quad \text{and} \quad \frac{\dot{\varphi}_1(t_1)}{\|\dot{\varphi}_1(t_1)\|}^T \frac{\dot{\varphi}_2^\perp(0)}{\|\dot{\varphi}_2^\perp(0)\|} = 0.$$

6. $\varphi_2^\perp$ and φ_2 intersect each other, at least for ϱ sufficiently small. Denote $\bar{\varrho} = \sup\{\varrho : \varphi_2^\perp \cap \varphi_2 \neq \emptyset\}$. Consider the point of intersection $u_2 = \varphi_2(t_2) = \varphi_2^\perp(t_2')$ between the two curves and let ϕ be the angle between the two curves at point u_2.

Finally, let $R(\theta, \varrho)$ be the geodesic quadrilateral joining points u_{goal}, u_1, u_2, u_1', u_{goal} through the geodesics φ_1, $\varphi_2^\perp$, φ_2, $\varphi_1^\perp$.

Observe that, for small enough ϱ, the point u_2 belongs to some neighborhood of u_{goal}. Moreover, the angle ϕ between $\varphi_2^\perp$ and φ_2 at u_2, $d(u_1, u_2)$ and $d(u_1', u_2)$ depend continuously on ϱ, in particular

$$\lim_{\varrho \mapsto 0} d(u_1, u_2) = \lim_{\varrho \mapsto 0} d(u_1', u_2) = 0 \tag{9}$$

and

$$\lim_{\varrho \mapsto 0} \phi = \frac{\pi}{2}. \tag{10}$$

Next we describe the trace of the geodesic quadrilateral $R(\theta, \varrho)$ on the plane (see fig.2), which is comprised of 4 straight segments of length $d(u_0, u_1) = \varrho, d(u_1, u_2), d(u_2, u_1')$ and $d(u_1', u_0) = \varrho$, respectively, and angle $\theta, \theta + \frac{\pi}{2}, \theta + \frac{3\pi}{2} - \phi$ and $\theta - \phi$.

Clearly, any geodesic quadrilateral is an element of the group $\mathcal{L}$ described in the previous section and, denoting $d(u_1, u_2) = \varrho_1$ and $d(u_2, u_1') = \varrho_1'$, we have that its action on the rolling object configuration $\Xi(R(\theta,\varrho)) = (v, \Delta\psi)$ is given by

$$v = e^{i\theta}\left(\varrho + \varrho_1 e^{i\frac{\pi}{2}} + \varrho_1' e^{i\frac{3\pi}{2}-\phi} + \varrho e^{-i\phi}\right), \tag{11}$$

and (using the Gauss-Bonnet theorem)

$$\Delta\psi = \left(3\frac{\pi}{2} + \phi\right) - 2\pi = -\frac{\pi}{2} + \phi. \tag{12}$$

5 Proof of the Local–Local Property Using Geodesic Quadrilaterals

First we show that the second step of the proposed algorithm verifies the local property. By equation (10) and (12) we can fix $\varrho \leq \bar{\varrho}$ such that $\Delta\psi \leq \frac{\varepsilon_\psi}{2}$. Then the trajectory u on the object surface is the geodesic quadrilateral $R(\varrho,\theta)$ repeated $N = \left[\frac{|\psi_1-\psi_2|}{\Delta\psi}\right]$ times. Clearly $u \subset B(u_{goal}, \delta_u)$ with $\delta_u = C'\varrho$ for some constant C'. As to locality of Step 2 in the plane, observe first that $v \in \mathbb{R}^2$ with $\Xi(R(\varrho,\theta)) = (v, \Delta\psi)$ is such that

$$\|v\| \leq C\varrho^2 \tag{13}$$

for some constant C. Indeed from equation (11) we obtain

$$\|v\|^2 = (\varrho - \varrho_1' \sin\phi + \varrho\cos\phi)^2 + (\varrho_1 - \varrho_1'\cos\phi - \varrho\sin\phi)^2$$

and, being $\phi = \frac{\pi}{2} + O(\varrho)$, $\frac{\varrho_1}{\varrho} = 1 + O(\varrho)$, and $\frac{\varrho_1'}{\varrho} = 1 + O(\varrho)$, we immediately obtain (13). Moreover, we have the following facts:

1.

$$\Delta\psi = \int_\Omega K du\ dv \geq K_{min}\varrho^2$$

where Ω is the region bounded by the geodesic quadrilateral and K and K_{min} are respectively the Gaussian curvature and the minimum of the Gaussian curvature in the closed region Ω, and

2. let $V = \sum_{n=0}^{N} v e^{in\Delta\psi}$ be the total displacement on the plane; then $\|V\| < N\|v\|$

From equation (13) it follows that

$$\|V\| \leq \frac{|\psi_1 - \psi_{goal}|}{\Delta\psi} C\varrho^2 \leq \frac{|\bar{\delta}_\psi|}{K_{min}\varrho^2} C\varrho^2 \leq \frac{|\bar{\delta}_\psi|}{K_{min}} C$$

Then we have that the trajectory x on the plane is such that $x \subset B(x_1, \delta_x)$ with $\delta_x = C\frac{|\bar{\delta}_\psi|}{K_{min}} = C'\bar{\delta}_\psi$, $C' = \frac{C}{K_{min}}$.

There remains to show that the local–local property also holds for the third step 3 of the proposed algorithm. To prove this, it is sufficient to find $\varrho \leq \bar{\varrho}$ such that $\Xi(R(\varrho,\theta))$ verifies equation (8). By equation (13) it is sufficient to choose $\varrho \leq \min\{C'\sqrt{\varepsilon_x}, \bar{\varrho}\}$, with C' some positive constant.

On the object surface the trajectory u is entirely contained in a neighborhood $B(u_{goal}, \delta_u)$ with $\delta_u = C'\varrho$ for

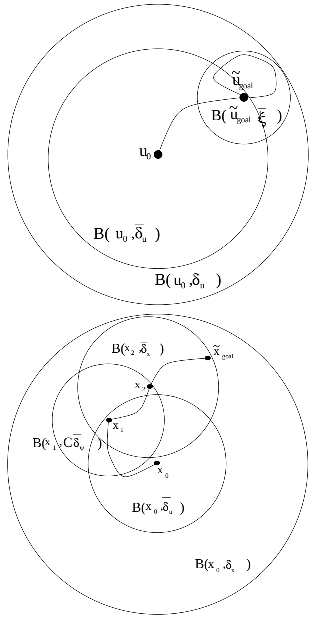

Figure 3: The complete trajectory of the contact point on the object surface (top) and on the plate (bottom) provided by the geodesic quadrilateral algorithm ($\bar{\zeta} = C\varrho_{min}$).

some constant C'. Moreover, along the trajectory, by equation (10),

$$|\psi(t)| \leq |\Delta\psi_1| + \Delta\psi_2| = |\frac{\pi}{2} - \phi_1| + |\frac{\pi}{2} - \phi_2| = C'\varrho$$

for some constant C'. Then for the trajectory ψ of the orientation it holds $\psi \in B(\psi_{goal}, \delta_\psi)$ with $\delta_\psi = C'\varrho$.

Finally for the trajectory x on the plane, a suitable combination of l_1 and l_2 can be found such that $x \in B(x_2, \delta_x)$ with $\delta_x = \bar{\delta}_x$.

Finally, let C be a constant bigger than all the constants C' found above, then we have that the parameters of equation (2) which hold for the global trajectory through steps 1,2, and 3 are (see fig.3) as follows:

$$\begin{aligned}
\delta_u &= \bar{\delta}_u + C\varrho_{min} \\
\delta_\psi &= C\bar{\delta}_u + \bar{\delta}_\psi + C\varrho_{min} \\
\delta_x &= \bar{\delta}_u + \bar{\delta}_x + C\bar{\delta}_\psi
\end{aligned}$$

where ϱ_{min} is the minimum among the parameters ϱ of Steps 1,2 and 3. Clearly any $\varrho \leq \varrho_{min}$ satisfy the local and steering properties of the algorithm and equation (5).

To illustrate the results of the above described algorithm, we report in fig.4 the solution obtained for the problem of planning rolling motions for an ellipsoid (with principal axes of length 30, 30, and 20 *cm*) within a corridor of width 55 *cm* and height 25 *cm*.

6 Conclusion

In this paper we have considered the problem of planning motions of objects rolling on a planar surface, and provided an algorithm that allows to deal

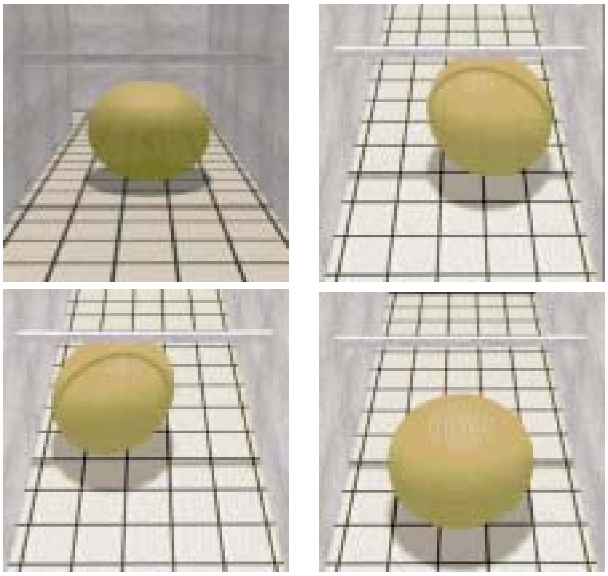

Figure 4: Four frames of the animation for the planning problem described in the text. The whole video can be downloaded from ftp://131.114.28.35/pub/uploads

with obstacles in the workspace in a systematic and effcient manner. The topological properties proved for the algorithm are a desirable characteristic also in the perspective of an iterative application for stabilization (cf. [6]), which is, as of today, an important and challenging open problem.

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